THE DERIVE – NEWSLETTER #123

THE BULLETIN OF THE

USER GROUP

+ CAS-TI

Contents:

September 2021

The **Risk Wiki** is a risk analysis manual and guideline that provides a good understanding of risk analysis techniques, tips and tricks, equations, uses and software tools.

In the right sidebar you will find a menu to browse the 1000 pages Risk Wiki.

Risk Wiki   https://www.vosesoftware.com/riskwiki/
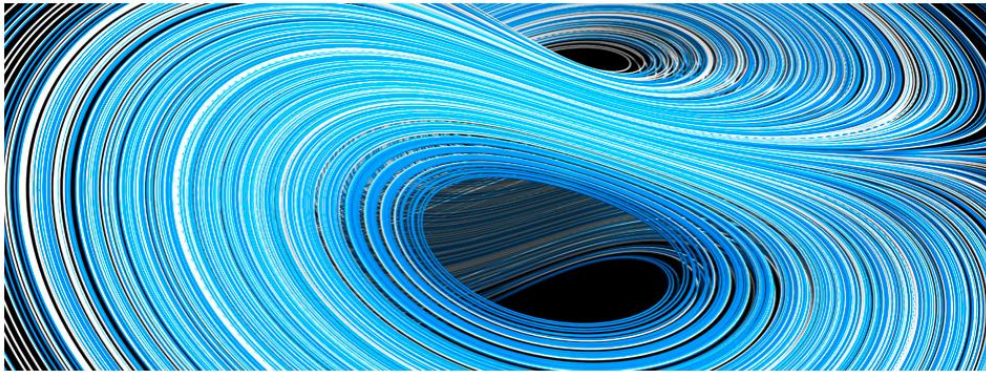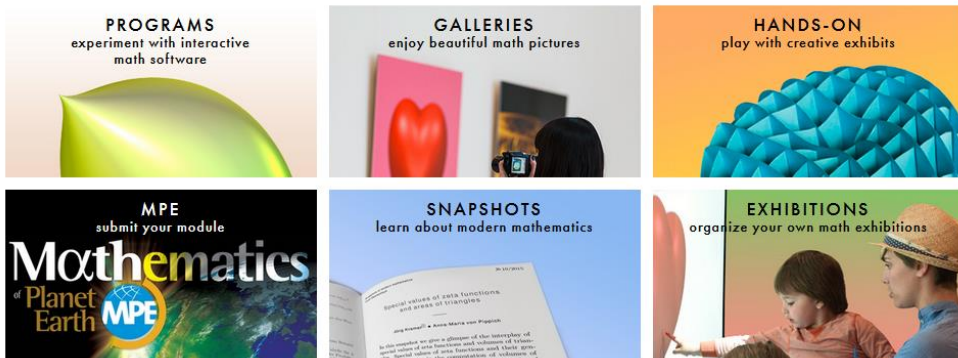
Generalized trapezoidal ogive curves for fatality rate modeling
https://www.sciencedirect.com/science/article/pii/S2590054420300245?dgcid=rss_sd_all

Explore the IMAGINARY – website:

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https://www.imaginary.org/

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https://momath.org/

Dear DUG-Members,

I am glad to be able to offer our DNL#123 in time. First of all I want to draw your attention to the links I have presented on the info page. The Imaginary website is really worth exploring. I came across Risk Wiki during my compilation of probability distributions. It is almost unbelievable how many distributions there are. With the second part of the "Survey ..." (starting on page 29), the topic is far from over. There are 15 to 20 more distributions to be discussed. Felipe Martinez Vargas' extensive *Derive* file PDF Y CDF.dfw is a very valuable support. For this issue, I have found the generalization of the trapezoidal distribution to be particularly challenging and interesting. The use of a CAS proves to be a great relief.

Wolfgang Alvermann entertains us again with a geometrical contribution. His treatment of n-circles addresses several fields of school mathematics and combines them into an exciting unity.

Heinrich Ludwig invites us to shuffle cards. "Skilled" card players can help themselves with the "shuffle". Heinrich hands over this work to a program.

The second part of Michel Beaudin's *Résumé* has again been left in its original French version. Translations into English and German will be offered additionally (in MTH123.zip).

Finally, I must admit to a serious error: In the last DNL I presented a program for the TI-Nspire to create and to draw polygons. I claimed that this can only be done using PYTHON. Shame on me! Of course, it can be done without. You only have to store the coordinates of the corners in two lists and then connect the points to the polygon using function *DrawPoly* (page 28).

Please also note the possibility to download the videos of the lectures of the "Education Session" of this year's ACA21 (link on page 21).

I remain with best regards

always Yours Josef

Liebe DUG-Mitglieder,

ich bin froh, Euch den DNL#123 rechtzeitig anbieten zu können. Da will ich Euch gleich auf die Links aufmerksam machen, die ich auf der Info-Seite zusammengestellt habe. Die Imaginary-Webseite ist es wirklich wert, erkundet zu werden. Auf Risk Wiki bin ich im Rahmen meiner Zusammenstellung der Wahrscheinlichkeitsverteilungen gestoßen. Es ist ja geradezu unglaublich, wie viele Verteilungen es gibt. Mit dem zweiten Teil des „Survey …" (ab Seite 29) ist das Thema noch lange nicht abgeschlossen. Da gibt es noch 15 bis 20 weitere Verteilungen zu besprechen. Felipe Martinez Vargas' umfangreiche *Derive*-Datei PDF Y CDF.dfw ist eine sehr wertvolle Unterstützung. Für diese Ausgabe habe ich die Verallgemeinerung der Trapezverteilung als besonders herausfordernd und interessant gefunden. Dabei erweist sich der Einsatz eines CAS als eine große Erleichterung.

Wolfgang Alvermann unterhält uns wieder mit einem geometrischen Beitrag. Seine Behandlung der n-Kreise spricht einige Gebiete der Schulmathematik an und verbindet sie zu einer spannenden Einheit.

Heinrich Ludwig lädt uns zum Kartenmischen ein. „Geschickte" Kartenspieler können sich ja beim „Mischen" helfen. Heinrich überlässt diese Arbeit einem Programm.

Der zweite Teil von Michel Beaudins *Résumé* wurde wieder in seiner französischen Originalfassung belassen. Übersetzungen ins Englische und Deutsche werden zusätzlich angeboten (in MTH123.zip).

Schließlich muss ich noch einen schweren Fehler eingestehen: Im letzten DNL wurde ein Programm für den TI-Nspire zur Erzeugung und zum Zeichnen von Polygonen vorgestellt. Da habe ich behauptet, dass sich das nur mit PYTHON machen lässt. Schande über mein Haupt! Natürlich geht das auch ohne. Man muss nur die Koordinaten der Ecken in Listen ablegen und dann die Punkte über *DrawPoly* zum Polygon verbinden (Seite 28).

Beachtet bitte auch die Möglichkeit zum Download der Videos der Vorträge der „Education Session" der heurigen ACA21 (Link auf Seite 21).

Mit besten Grüßen verbleibe ich wie immer

Euer Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: December 2021

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER

Wonderful World of Pedal Curves, J. Böhm, AUT

Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT

Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT

Logos of Companies as an Inspiration for Math Teaching

Exciting Surfaces in the FAZ

BooleanPlots.mth, P. Schofield, UK

Old traditional examples for a CAS – What´s new? J. Böhm, AUT

Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ

Tutorials for the NSpireCAS, G. Herweyers, BEL

Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA

Another Approach to Taylor Series, D. Oertel, GER

Charge in a Magnetic Field, H. Ludwig, GER

More Applications of TI-Innovator[TM] Hub and TI-Innovator[TM] Rover

Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT

A Collection of Special Problems, W. Alvermann, GER

DERIVE Bugs? D. Welz, GER

Tweening & Morphing with TI-NspireCX-II-T, J. Böhm. AUT

The Gap between Poor and Rich, J. Böhm, AUT

More functions from M. Myers and from Bhuvanesh's Mathtools-library

Double-Die-Encryption – Doppelwürfelverschlüsselung

TaxiCab Conics, Two alternate Approaches to Conics, R. Haas, USA

QR-Code light

Sparse Matrices,153 is another Special Number, and others

n-Kreis Figuren – *n*-Circle Figures

Wolfgang Alvermann, Germany

n-Kreis Figur[1] n / Circle Figure

„Besondere Kreisfiguren liegen auch vor, wenn man um die Eckpunkte regelmäßiger *n*-Ecke Kreise mit einem Radius zeichnet, der gleich dem Abstand zweier benachbarter Eckpunkte ist. Allerdings ist die Bestimmung der Flächeninhalte nicht immer leicht. Die folgenden Abbildungen zeigen *n*-Kreis-Figuren für *n* = 5, 6, 7."

"Special circular figures are also present if circles with a radius equal to the distance between two adjacent vertices are drawn around the vertices of regular *n* vertices. However, calculation of the area is not always easy. The following figures show *n*-circle figures for *n* = 5, 6, 7. "

[1] Heinz Klaus Strick: Mathematik ist wunderwunderschön (Seite 64), Springer 2019

Aufgabe / Task:
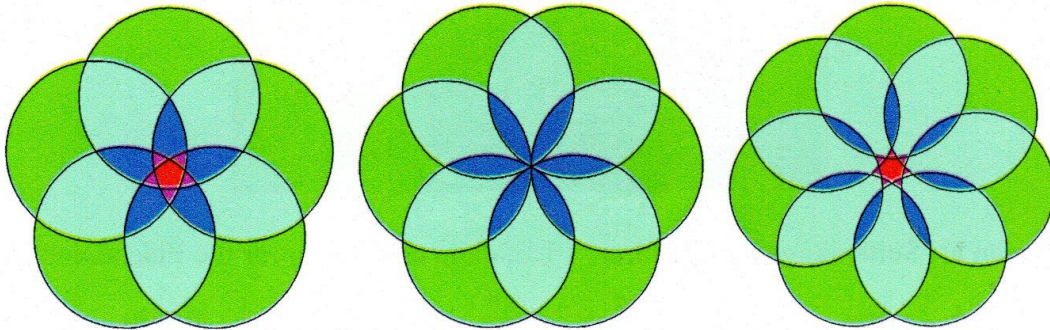
Berechnen Sie jeweils die Größe der Gesamtfläche der dargestellten *n*-Kreis Figuren.

Calculate the total area of each of the *n*-circle figures shown.

Vorüberlegungen / Preliminary considerations:

Zur Lösung der Aufgaben kann u. a. die Fläche regelmäßiger Vielecke benutzt werden.
Anhand der dargestellten Figur kann man folgendes herleiten:

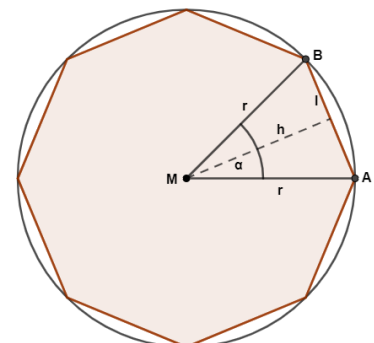To solve the tasks, among others, the area of regular polygons can be used.
On the basis of the figure shown, you can deduce the following:

<u>Seitenlänge *l* / Side *l*</u>: $l(n) = 2 \cdot \sin\left(\dfrac{\pi}{n}\right) \cdot r$

<u>Höhe *h* / Altitude *h*</u>: $h(n) = \dfrac{n}{2} \cdot \cos\left(\dfrac{\pi}{n}\right) \cdot r$

<u>Fläche *A* / Area *A*</u>:

$$A(n) = \frac{1}{2} \cdot l \cdot h \cdot n$$

$$A(n) = n \cdot \frac{1}{2} \cdot 2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right) \cdot r^2 \quad \Rightarrow \quad \boxed{A(n) = \frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right) \cdot r^2}$$

6-Kreis-Figur, Lösung 1 / 6-Circle Figure, Solution 1

Um die Eckpunkte des schwarzen regelmäßigen Sechsecks mit Radius *r* sind sechs Kreise mit ebenfalls Radius *r* gezeichnet. Die Schnittpunkte dieser Kreise ergeben ein weiteres regelmäßiges Sechseck, dessen Radius *R* noch bestimmt werden muss. Die Gesamtfläche ergibt sich dann als Summe aus diesem größeren Sechseck und sechs Kreisabschnitten mit Radius *r* und Winkel $\alpha = 120° = \frac{2}{3}\pi$ und der Formel $A = \frac{r^2}{2} \cdot (\alpha - \sin(\alpha))$.

Around the vertices of the black regular hexagon with radius *r* six circles with same radius *r* are drawn. The intersection points of these circles result in another regular hexagon whose radius *R* still has to be determined. The total area is the sum of this larger hexagon and six circular segments with radius *r* and angle $\alpha = 120° = \frac{2}{3}\pi$ and the formula $A = \frac{r^2}{2} \cdot (\alpha - \sin(\alpha))$
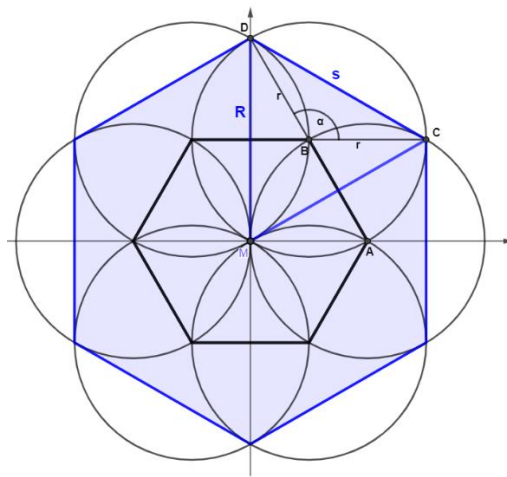
Der Radius R / **Radius *R*:**

Der Punkt D liegt auf einem Kreis um $B = \left(\frac{r}{2} / \frac{\sqrt{3}}{2}r\right)$ mit Radius *r* und dem *x*-Wert *x* = 0.

Point D lies on a circle with center $B = \left(\frac{r}{2} / \frac{\sqrt{3}}{2}r\right)$ and radius *r* with *x* = 0.

$$\left.\begin{array}{l}\left(x - \frac{r}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}r\right)^2 = r^2 \\ x = 0\end{array}\right\} \Rightarrow D = \left(0 / \sqrt{3} \cdot r\right) \qquad \boxed{R = \sqrt{3} \cdot r}$$

Die Teilflächen / **the partial areas:**

Das Sechseck / Hexagon: $\qquad A_{6-Eck} = \frac{6}{2} \cdot \sin(60°) \cdot 3r^2 \qquad A_{6-Eck} = \frac{9}{2}\sqrt{3} \cdot r^2$

Die Kreisabschnitte / Segments:

$$A_{Ka} = 6 \cdot \frac{r^2}{2} \cdot \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) \qquad A_{Ka} = \frac{r^2}{2} \cdot \left(4\pi - 3 \cdot \sqrt{3}\right)$$

Die Gesamtfläche / **Total area:**

$$A_g = \frac{9}{2}\sqrt{3} \cdot r^2 + \frac{r^2}{2} \cdot \left(4\pi - 3 \cdot \sqrt{3}\right) \qquad \boxed{A_g = \left(2\pi + 3 \cdot \sqrt{3}\right) \cdot r^2 \approx 11.479 \cdot r^2}$$

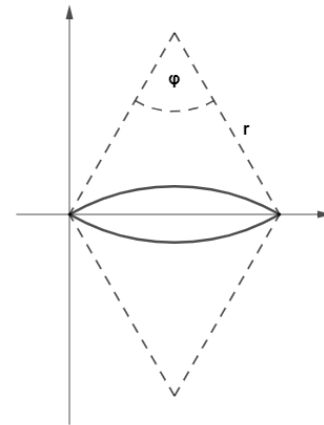### 6-Kreis-Figur, Lösung 1 / 6-Circle Figure, Solution 2

In dieser Lösung werden die farbigen Flächen betrachtet.

In this solution, the colored surfaces are considered.

Die blaue Fläche / the blue area

Die blaue Fläche besteht aus 12 Kreisabschnitten mit Radius r und Winkel $\varphi = 60° = \frac{\pi}{3}$.

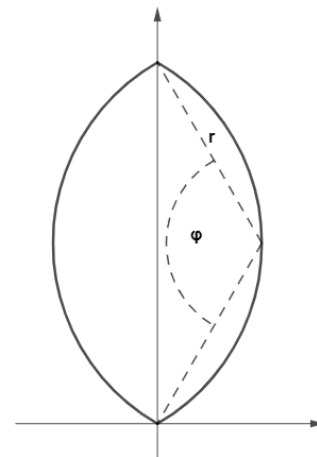The blue area consists of 12 segments with radius r and angle $\varphi = 60° = \frac{\pi}{3}$.

Mit der Formel für Kreisabschnitte (formula for segment:) $A = \frac{r^2}{2} \cdot \left(\varphi - \sin(\varphi)\right)$ erhält man:

$$A_{bl} = 12 \cdot \frac{r^2}{2} \cdot \left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)$$
$$A_{bl} = 6 \cdot r^2 \cdot \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \quad \rightarrow \quad \boxed{A_{bl} = r^2 \cdot \left(2\pi - 3 \cdot \sqrt{3}\right)}$$

Die hellblaue Fläche / the light blue area

Eine hellblaue Fläche besteht aus zwei Kreisabschnitten mit Radius r und dem Winkel $\varphi = 120° = \frac{2}{3}\pi$ vermindert um 1/3 von A_{bl}.

The light blue area consists of two segments (radius r and angle $\varphi = 120° = \frac{2}{3}\pi$ minus one third of A_{bl}.

$$A_{hgr} = 6 \cdot \left[r^2 \cdot \left(\frac{2}{3}\pi - \sin\left(\frac{2}{3}\pi\right)\right) - \frac{r^2}{3} \cdot \left(2\pi - 3 \cdot \sqrt{3}\right) \right]$$

$$A_{hgr} = 6 \cdot r^2 \cdot \left(\frac{2}{3}\pi - \frac{\sqrt{3}}{2}\right) - 2r^2 \cdot \left(2\pi - 3 \cdot \sqrt{3}\right) \qquad \boxed{A_{hbr} = 3 \cdot \sqrt{3} \cdot r^2}$$

$$A_{hgr} = r^2 \cdot \left[\left(4\pi - 3 \cdot \sqrt{3}\right) - \left(4\pi - 6 \cdot \sqrt{3}\right)\right]$$

Die grüne Fläche / the green area

Eine grüne Teilfläche besteht aus einem Kreis vermindert um ein Drittel der hellblauen Fläche und vermindert um die Hälfte der blauen Fläche.

One green subarea consists of a circle reduced by one third of the light blue area and reduced by half of the blue area.

$$A_{gr} = 6 \cdot \left[\pi \cdot r^2 - \sqrt{3} \cdot r^2 - \frac{r^2}{2} \cdot \left(2\pi - 3 \cdot \sqrt{3} \right) \right] \Rightarrow \boxed{A_{gr} = 3 \cdot \sqrt{3} \cdot r^2}$$

Die Gesamtfläche / total area

Die Addition $A_{bl} + A_{hbl} + A_{gr}$ ergibt wieder:
The sum of $A_{bl} + A_{hbl} + A_{gr}$ results again in:

$$\boxed{A_g = \left[2\pi + 3 \cdot \sqrt{3} \right] r^2 \approx 11.479 \cdot r^2}$$

5-Kreis-Figur / 5 Circle Figure

Bei der Berechnung dieser Fläche gilt dasselbe wie bei der 6-Kreis-Figur bei der Lösung 1.

Zu ermitteln sind:

➤ $r_1 = \overline{AB}$, $\overline{CD}$ und α zur Berechnung der Kreisabschnitte A_{Ka}

➤ r_2 zur Bestimmung des 5-Ecks

When calculating this area, the same applies as for the 6-circle figure in solution 1

To be determined are:

➤ $r_1 = \overline{AB}$, $\overline{CD}$ and α for calculating the circular segments A_{Ka}

➤ r_2 for finding the heptagon
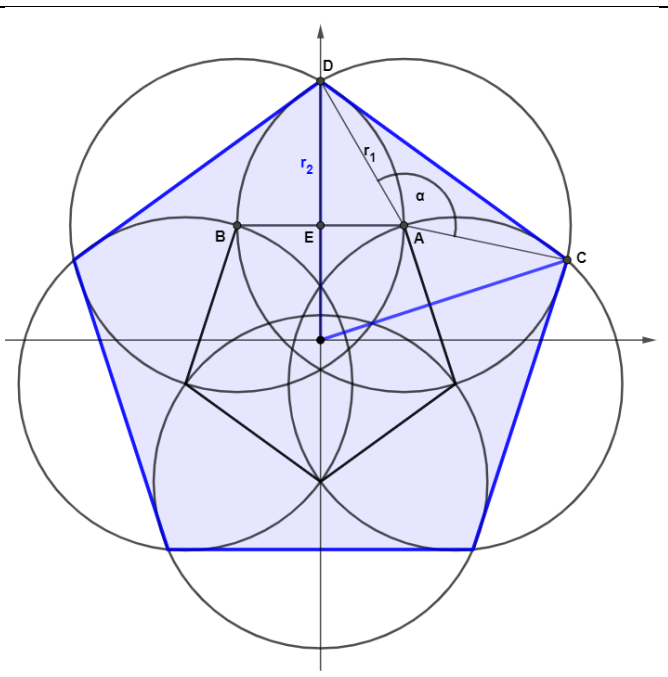
Dazu benötigt man zunächst die Koordinaten der Punkte von A und B.

We first need the coordinates of the points of A and B: $A = \left(r \cdot \cos(54°) \, / \, r \cdot \sin(54°) \right)$

$$A = \left(\frac{\sqrt{10 - 2 \cdot \sqrt{5}}}{4} \cdot r \, / \, \frac{1 + \sqrt{5}}{4} \cdot r \right)$$

$$B = \left(-\frac{\sqrt{10 - 2 \cdot \sqrt{5}}}{4} \cdot r \, / \, \frac{1 + \sqrt{5}}{4} \cdot r \right)$$

Daraus ergibt sich r_1 zu / This leads to r_1: $r_1 = 2 \cdot x(A)$: $\quad r_1 = \frac{\sqrt{10 - 2 \cdot \sqrt{5}}}{2} \cdot r$

r_2 erhält man aus $y(A)$ und dem Dreieck ADE:

$$r_2 = y(A) + \sqrt{r_1^2 - \left(\frac{r_1}{2}\right)^2}$$

r_2 follows from $y(A)$ and triangle ADE:

$$r_2 = \left(\sqrt{5} + 1 + \sqrt{6 \cdot (5 - \sqrt{5})}\right) \cdot \frac{r}{4}$$

Die 5-Eckseite $\overline{CD} = L$ ergibt sich zu

$$L = r_2 \cdot \sqrt{2 \cdot \left(1 - \cos\left(\frac{2\pi}{5}\right)\right)}$$

Side $\overline{CD} = L$ of the pentagon results in

$$L(r) = \frac{\left(\sqrt{5} + 1\right) \cdot \left(\sqrt{2 \cdot (5 - \sqrt{5})} + 2 \cdot \sqrt{3} \cdot (5 - \sqrt{5})\right)}{8} \cdot r$$

Mit dem Kosinussatz erhält man aus dem Dreieck ACD / Applying the cosine rule in ΔACD:

$$\alpha = \cos^{-1}\left(1 - \frac{L(r)^2}{2 \cdot r_1(r)^2}\right), \quad \text{Mit den oben angegeben Formeln ergibt sich } \underline{\alpha = 132°}$$

Mit den Formeln für ein *n*-Eck und für Kreisabschnitte erhält man die Gesamtfläche mit:

Finally, the total area (pentagon + circular segments) is:

$$A_g = \frac{5}{2} \cdot \sin\left(\frac{2\pi}{5}\right) \cdot r_2^2 + 5 \cdot \frac{r_1^2}{2} \cdot \left(132° - \sin(132°)\right)$$

$$\boxed{A_g \approx 13.3292 \cdot r^2}$$

Ein TI-Nspire-Protokoll:

Die 5−Kreis−Figur

Benötigt werden der Punkt A, der Abstand AB = r_1, der Radius R = r_2 des großen Fünfecks und der Winkel α.

$$\mathbf{xa}(r) := r \cdot \cos\left(\frac{3 \cdot \pi}{10}\right) : \mathbf{ya}(r) := r \cdot \sin\left(\frac{3 \cdot \pi}{10}\right) : \mathbf{r1}(r) := 2 \cdot \mathbf{xa}(r) : \mathbf{r1}(r) \ \blacktriangleright \ \frac{\sqrt{-2 \cdot (\sqrt{5} - 5)} \cdot r}{2}$$

$$\mathbf{r2}(r) := \mathbf{ya}(r) + \sqrt{\left(\mathbf{r1}(r)\right)^2 - \left(\frac{\mathbf{r1}(r)}{2}\right)^2} : \mathbf{r2}(r)|r > 0 \ \blacktriangleright \ \left(\frac{\sqrt{-6 \cdot (\sqrt{5} - 5)}}{4} + \frac{\sqrt{5} + 1}{4}\right) \cdot r$$

Die Fläche des 5−Ecks

$$\mathbf{a5}(r) := \frac{5}{2} \cdot \sin\left(\frac{2 \cdot \pi}{5}\right) \cdot \left(\mathbf{r2}(r)\right)^2 : \mathbf{a5}(r)|r > 0 \ \blacktriangleright \ \left(\frac{5 \cdot (\sqrt{5} + 1) \cdot \sqrt{60}}{32} - \frac{5 \cdot (\sqrt{5} - 9) \cdot \sqrt{2 \cdot (\sqrt{5} + 5)}}{32}\right) \cdot r^2$$

$$\mathbf{a5}(r)|r > 0 \ \blacktriangleright \ 7.93719 \cdot r^2$$

Seitenlänge l und Winkel α

$$\mathbf{l}(r) := \mathbf{r2}(r) \cdot \sqrt{2 \cdot \left(1 - \cos\left(\frac{2 \cdot \pi}{5}\right)\right)} : \alpha := \cos^{-1}\left(1 - \frac{\left(\mathbf{l}(r)\right)^2}{2 \cdot \left(\mathbf{r1}(r)\right)^2}\right)|r > 0 \ \blacktriangleright \ 2.30383$$

Die Kreisabschnitte/Gesamtfläche: $\mathbf{aka}(r) := \frac{5}{2} \cdot \left(\mathbf{r1}(r)\right)^2 \cdot \left(\alpha - \sin(\alpha)\right) : \mathbf{aka}(r)|r > 0 \ \blacktriangleright \ 5.39205 \cdot r^2$

Die Fläche in allgemeiner Form mit dem TI-Nspire-CAS:

The general formula for the total area derived with TI-NspireCAS.

$$\mathbf{a5}(r)+\mathbf{aka}(r)|r>0$$

$$\blacktriangleright \left(\frac{-5\cdot\left(\sqrt{5}-5\right)\cdot\sin^{-1}\left(\dfrac{\left(\sqrt{5}+1\right)\cdot\sqrt{-6\cdot\left(\sqrt{5}-5\right)}-2\cdot\left(\sqrt{5}-1\right)}{16}\right)}{8} - \frac{5\cdot\pi\cdot\left(\sqrt{5}-5\right)}{16} \right.$$

$$\left. + \frac{5\cdot\left(\sqrt{5}-5\right)\cdot\sqrt{\sqrt{-6\cdot\left(\sqrt{5}-5\right)}-\sqrt{5}+7}}{32} - \frac{5\cdot\left(\sqrt{5}-9\right)\cdot\sqrt{2\cdot\left(\sqrt{5}+5\right)}}{32} + \frac{5\cdot\left(\sqrt{5}+1\right)\cdot\sqrt{60}}{32} \right)\cdot r^2$$

Ganz schön unhandlich! **Pretty bulky!**

Eine allgemeine Lösung für die *n*-Kreis-Figur

A general solution for the *n*-circle figure

Anhand der nebenstehenden Grafik soll eine allgemeine Formel für die Fläche einer *n*-Kreis-Figur mit Hilfe des TI-NspireCAS entwickelt werden.

Dazu genügt es, den Punkt A in Abhängigkeit von n derart fest-zulegen, dass $\overline{AB} = r_1(n)$ parallel zur *x*-Achse liegt; B entsteht durch Spiegelung von A an der *y*-Achse.

Using the adjacent graph, a general formula for the area of an *n*-circle figure will be developed using TI-NspireCAS.
For this purpose, it is sufficient to define the point A in dependence of *n* such that $\overline{AB} = r_1(n)$ is parallel to the *x*-axis; point B is created by reflection on the *y*-axis.
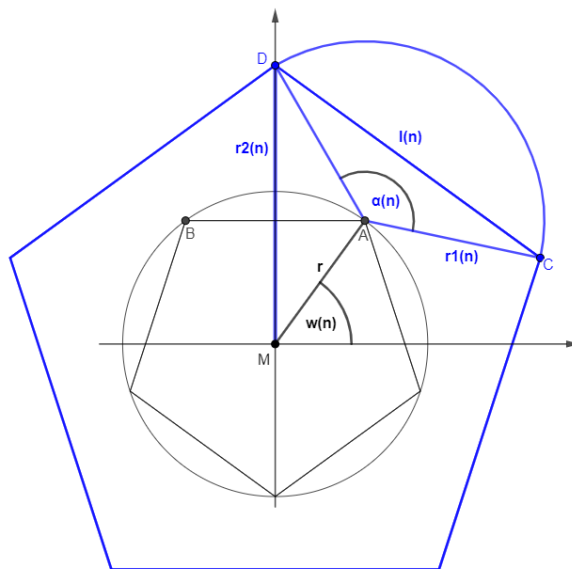
Der Winkel $w(n)$ ergibt sich zu $w(n) = \dfrac{n-2}{2n}\cdot\pi$. Daraus folgt: $A = \left[r\cdot\cos(w(n))/r\cdot\sin(w(n))\right]$.

Angle $w(n) = \dfrac{n-2}{2n}\cdot\pi$ and subsequently point $A = \left[r\cdot\cos(w(n))/r\cdot\sin(w(n))\right]$.

Das TI-Nspire Protokoll / The TI-Nspire protocol:

n-Kreis-Figuren

Definition des Punktes A über Radius und Winkel

$$\mathbf{w}(n) := \frac{n-2}{2 \cdot n} \cdot \pi : \mathbf{xa}(n) := r \cdot \cos(\mathbf{w}(n)) : \mathbf{ya}(n) := r \cdot \sin(\mathbf{w}(n))$$

Die erforderlichen Radien

$$\mathbf{r1}(n) := 2 \cdot \mathbf{xa}(n) : \mathbf{r2}(n) := \mathbf{ya}(n) + \sqrt{(\mathbf{r1}(n))^2 - \left(\frac{\mathbf{r1}(n)}{2}\right)^2}$$

Die Länge l und der Winkel α

$$\mathbf{l}(n) := 2 \cdot \mathbf{r2}(n) \cdot \sin\left(\frac{\pi}{n}\right) : \alpha(n) := \cos^{-1}\left(1 - \frac{(\mathbf{l}(n))^2}{2 \cdot (\mathbf{r1}(n))^2}\right)$$

n-Eck-Fläche, Fläche der Kreisabschnitte und Gesamtfläche

$$\mathbf{an}(n) := \frac{n}{2} \cdot \sin\left(\frac{2 \cdot \pi}{n}\right) \cdot (\mathbf{r2}(n))^2 : \mathbf{aka}(n) := \frac{n}{2} \cdot (\alpha(n) - \sin(\alpha(n))) \cdot (\mathbf{r1}(n))^2 : \mathbf{ages}(n) := \mathbf{an}(n) + \mathbf{aka}(n)$$

Beispielrechnung

$$\mathbf{ages}(6)|r>0 \quad \blacktriangleright \quad 11.4793 \cdot r^2 \qquad \mathbf{ages}(5)|r>0 \quad \blacktriangleright \quad 13.3292 \cdot r^2$$

$$\mathbf{ages}(7)|r>0 \quad \blacktriangleright \quad 10.1445 \cdot r^2$$

This is the – again very bulky – general formula for the total sum of *n*-circle figures:

Die allgemeine – unhandliche – Formel

$$\mathbf{ages}(n)|r>0$$

$$\blacktriangleright \left(\frac{\left(\cos\left(\frac{\pi}{n}\right) + \left|\sin\left(\frac{\pi}{n}\right)\right| \cdot \sqrt{3}\right)^2 \cdot \sin\left(\frac{2 \cdot \pi}{n}\right)}{2} \right.$$

$$-\left(\sin\left(\frac{\pi}{n}\right)\right)^2 \cdot \left(\sqrt{-8 \cdot \left(\sin\left(\frac{\pi}{n}\right)\right)^2 \cdot \left(\cos\left(\frac{\pi}{n}\right)\right)^2 - 4 \cdot \left|\sin\left(\frac{\pi}{n}\right)\right| \cdot \left(2 \cdot \left(\sin\left(\frac{\pi}{n}\right)\right)^2 - 1\right) \cdot \cos\left(\frac{\pi}{n}\right) \cdot \sqrt{3} + 3} \right.$$

$$\left. \left. -2 \cdot \sin^{-1}\left(\frac{2 \cdot \left|\sin\left(\frac{\pi}{n}\right)\right| \cdot \cos\left(\frac{\pi}{n}\right) \cdot \sqrt{3} + 2 \cdot \left(\sin\left(\frac{\pi}{n}\right)\right)^2 - 1}{2}\right) - \pi \right) \right) \cdot n \cdot r^2 \; \triangle$$

Grenzwertbetrachtungen / Considerations about limits

$$n \to \infty \text{ daraus folgt} \begin{cases} w(n) \to \dfrac{\pi}{2} & x_a(n) \to 0 \quad y_a(n) \to r \\ r_1(n) \to 0 & r_2(n) \to r \quad l(n) \to 0 \\ a_n(n) \to \pi \cdot r^2 & a_{ka}(n) \to 0 \quad a_{ges}(n) \to \pi \cdot r^2 \end{cases}$$

Eine ausführliche Grenzwertbetrachtung für:
A more extended consideration for:
$$a_n(n) = \frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right) \cdot (r_2(n))^2$$

$$\lim_{n\to\infty}\left(\frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right)\right)$$
es handelt sich um den Fall "$0 \cdot \infty$" *Case "$0 \cdot \infty$"*

$$\lim_{n\to\infty}\left(\frac{\sin\left(\dfrac{2\pi}{n}\right)}{\dfrac{2}{n}}\right)$$
umgeformt ergibt sich der Fall "$\dfrac{0}{0}$" *Case "$\dfrac{0}{0}$"*

$$\lim_{n\to\infty}\left(\frac{\cos\left(\dfrac{2\pi}{n}\right) \cdot \dfrac{-2\pi}{n^2}}{\dfrac{-2}{n^2}}\right)$$
Regel von de l'Hopital: $\lim\limits_{n\to\infty}\left(\dfrac{f(n)}{g(n)}\right) = \lim\limits_{n\to\infty}\left(\dfrac{f'(n)}{g'(n)}\right)$

$$\boxed{\lim_{n\to\infty}\left(\frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right)\right) = \pi}$$

When Wolfgang sent this contribution, I was immediately inspired to try plotting these pretty figures using *Derive* and/or TI-Nspire.

In *Derive* one can shade areas by using inequalities and by combining them applying Boolean opera

tions. I will present two ways plotting the *n*-circles.

The 6-circle will be produced by defining the single parts of the "flower" separately. I start with a base hexagon (the blue one in Wolfgang's sketch on page 4) and the six circles. This gives the structure.

$$\text{mp} := \text{VECTOR}\left(\left[6 \cdot \cos\left(\frac{k \cdot \pi}{3}\right), 6 \cdot \sin\left(\frac{k \cdot \pi}{3}\right)\right], k, 0, 6\right)$$

$$\text{mp} := \begin{bmatrix} 6 & 0 \\ 3 & 3 \cdot \sqrt{3} \\ -3 & 3 \cdot \sqrt{3} \\ -6 & 0 \\ -3 & -3 \cdot \sqrt{3} \\ 3 & -3 \cdot \sqrt{3} \\ 6 & 0 \end{bmatrix}$$

$$\text{VECTOR}\left((x - \text{mp}_{k,1})^2 + (y - \text{mp}_{k,2})^2 = 36, k, 0, 6\right)$$
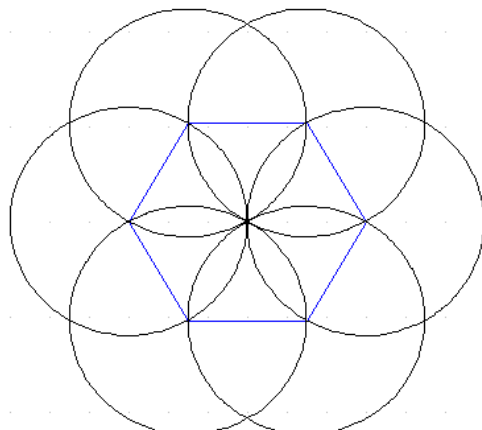
Then I define the areas within and outside of the circles (ki and ko):

$$ki := VECTOR\left((x - mp_{k,1})^2 + (y - mp_{k,2})^2 \le 36, k, 1, 6\right)$$

$$ko := VECTOR\left((x - mp_{k,1})^2 + (y - mp_{k,2})^2 \ge 36, k, 1, 6\right)$$

Next expression describes the green parts as a Boolean expression:

$$(ko_1 \wedge ko_3 \wedge ki_2) \vee (ko_2 \wedge ko_4 \wedge ki_3) \vee (ko_3 \wedge ko_5 \wedge ki_4) \vee (ko_4 \wedge ko_6 \wedge$$
$$ki_5) \vee (ko_5 \wedge ko_1 \wedge ki_6) \vee (ko_6 \wedge ko_2 \wedge ki_1)$$

Next two expressions give the blue colored areas:

$$(ki_1 \wedge ki_2 \wedge ko_6 \wedge ko_3) \vee (ki_2 \wedge ki_3 \wedge ko_1 \wedge ko_4) \vee (ki_3 \wedge ki_4 \wedge ko_2 \wedge ko_5)$$

$$(ki_4 \wedge ki_5 \wedge ko_3 \wedge ko_6) \vee (ki_5 \wedge ki_6 \wedge ko_4 \wedge ko_1) \vee (ki_6 \wedge ki_1 \wedge ko_5 \wedge ko_2)$$

The six red areas complete the figure:

$$(ki_1 \wedge ki_3) \vee (ki_2 \wedge ki_4) \vee (ki_3 \wedge ki_5) \vee (ki_4 \wedge ki_6) \vee (ki_5 \wedge ki_1) \vee (ki_6 \wedge ki_2)$$

I admit that this is not the most comfortable way (compare with the 5-circle figure next page), but it could be a fine exercise for applying Boolean operations.

This is the *Derive* plot:
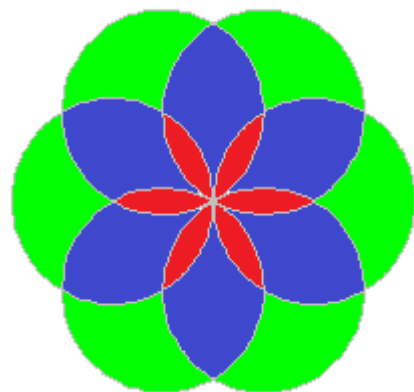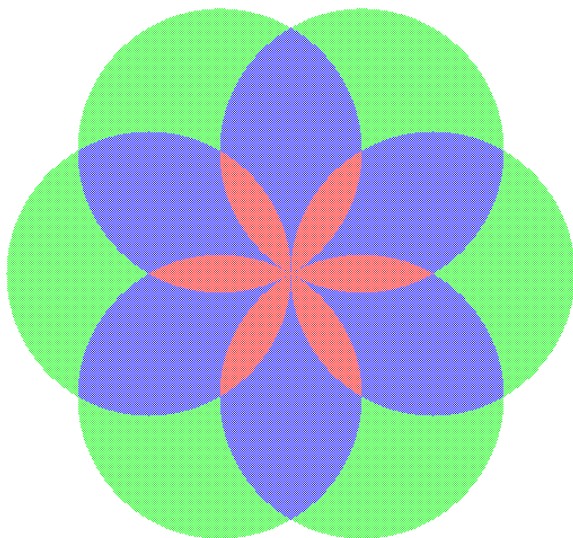
I used the "structure" from above and filled it using Paintbrush within the *Derive*-environment.

One could use a graphics program to color the figure, but this only half as funny!!

You can do this from within *Derive*!! This is a not so well know feature from *Derive*. See how this is done in Tania Koller's contribution in DNL#63!!

Derive und die 5-Kreis-Figur / The *Derive* treatment of the 5-Circle Figure

$$pp := \text{VECTOR}\left(\left[6 \cdot \cos\left(\frac{k \cdot 2 \cdot \pi}{5} - \frac{\pi}{2}\right), \; 6 \cdot \sin\left(\frac{k \cdot 2 \cdot \pi}{5} - \frac{\pi}{2}\right)\right], \; k, \; 1, \; 9\right)$$

$$\left(r1 := \left|pp_1 - pp_2\right|\right) = r1 := 3 \cdot \sqrt{(10 - 2 \cdot \sqrt{5})}$$

$$c := \text{VECTOR}\left((x - pp_{k,1})^2 + (y - pp_{k,2})^2 = r1^2, \; k, \; 1, \; 5\right)$$
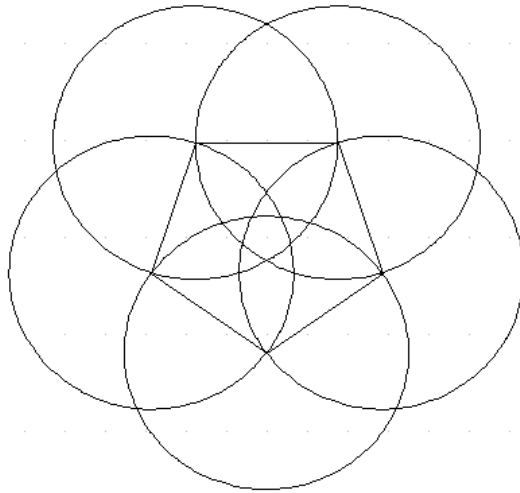
Green area

$$ci := \text{VECTOR}\left((x - pp_{k,1})^2 + (y - pp_{k,2})^2 \leq r1^2, \; k, \; 1, \; 9\right)$$

Blue area

$$\text{VECTOR}(ci_k \wedge ci_{k+1}, \; k, \; 1, \; 5)$$

Red area

$$\text{VECTOR}(ci_k \wedge ci_{k+2}, \; k, \; 1, \; 5)$$

Dark green area

$$\text{VECTOR}(ci_k \wedge ci_{k+3} \wedge \neg ci_{k+4}, \; k, \; 1, \; 5)$$

Light blue area

$$ci_1 \wedge ci_2 \wedge ci_3 \wedge ci_4 \wedge ci_5$$
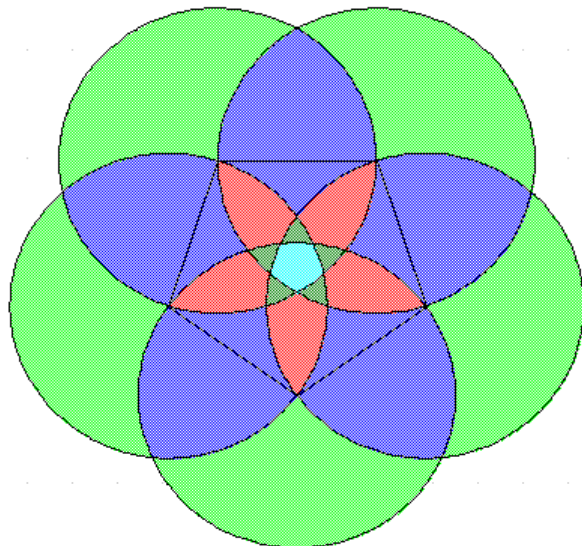
Résumé 1 – Part 2
Résolution symbolique, résolution numérique d'équations et de systèmes d'équations
Rappels et compléments en équations différentielles

1.8 Exemple d'une réponse « symbolique » Lorsqu'on considère une équation polynomiale de degré 5 ou plus, on peut toujours trouver, en arithmétique flottante, toutes les solutions. Soit, pour fixer les idées $x^5 + x + 3 = 0$. *Derive* retourne l'équation, à moins d'approximer. Sa fonction « solutions » retournant une matrice, le mode exact peut retourner une matrice vide!

```
        5
SOLVE(x  + x + 3 = 0, x)

 5
x  + x = -3

x = -0.4753807566 - 1.129701725·i ∨ x = -0.4753807566 + 1.129701725·i ∨ x = 1.041879539 - 0.8228703381·i ∨

 x = 1.041879539 + 0.8228703381·i ∨ x = -1.132997565
```

Figure 1.12

Maple: pour un polynôme, la structure « RootOf » permet d'utiliser chacune des racines.

> $solve(x^5 + x + 3 = 0, x);$

$$RootOf(\_Z^5 + \_Z + 3, index = 1), RootOf(\_Z^5 + \_Z + 3, index = 2), RootOf(\_Z^5 + \_Z + 3, index = 3), RootOf(\_Z^5 + \_Z + 3, index = 4), RootOf(\_Z^5 + \_Z + 3, index = 5)$$

Figure 1.13

1.9 Exemple et le gros bon sens Lorsqu'on considère une équation non polynomiale, alors les systèmes symboliques ne remplacent pas l'analyse du l'utilisateur et l'imagination et le gros bon sens sont toujours de mise. Il n'y a pas de formule pour résoudre l'équation $\sin x = 1 - \dfrac{x}{6}$. La calculatrice TI « avertit » l'utilisateur que certaines solutions seront peut-être oubliées :

⚠ $solve\left(\sin(x)=1-\dfrac{x}{6},x\right)$	$x=0.988567$ or $x=2.5236$ or $x=6.24272$ or $x=10.2002$ or $x=11.4336$
$exact\left(solve\left(\sin(x)=1-\dfrac{x}{6},x\right)\right)$	$6 \cdot \sin(x)+x=6$
$nSolve\left(\sin(x)=1-\dfrac{x}{6},x\right)$	0.988567
$nSolve\left(\sin(x)=1-\dfrac{x}{6},x=2\right)$	2.5236
$nSolve\left(\sin(x)=1-\dfrac{x}{6},x\right)|3<x<7$	6.24272
$nSolve\left(\sin(x)=1-\dfrac{x}{6},x,9,11\right)$	10.2002
$nSolve\left(\sin(x)=1-\dfrac{x}{6},x,11,\infty\right)$	11.4336

Figure 1.14

Un graphique de chacun des côtés de l'équation indique qu'il y a, en tout, 5 solutions. On voit donc de la Figure 1.14, que Nspire CX CAS a bel et bien trouvé (dans ce cas-ci) ces 5 solutions lorsque le système passe en mode approximatif. En mode exact, il ne retourne que l'équation sous forme équivalente. Et guider le « nsolve » peut être utile.

Regardons les réponses de *Maple* et de WolframAlpha dans la figure 1.15.

> $solve\left(\sin(x) = 1 - \dfrac{x}{6}, x\right);$

$$RootOf(6\sin(\_Z) - 6 + \_Z)$$

> $fsolve\left(\sin(x) = 1 - \dfrac{x}{6}, x\right);$

$$0.9885669797$$

> $fsolve\left(\sin(x) = 1 - \dfrac{x}{6}, x = 2\right);$

$$2.523600393$$
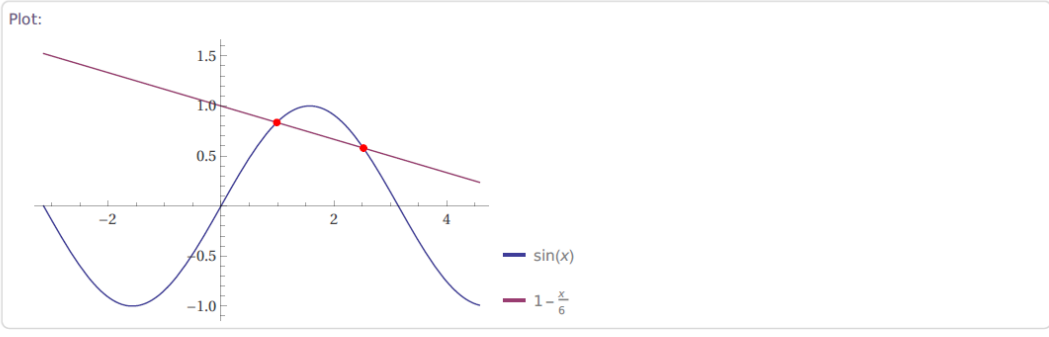
Figure 1.15

Comment réagit *Derive*? Au départ, une fenêtre contenant le graphe de chaque côté de l'équation indique bien 5 intersections au total, comme on l'indiquait tantôt:
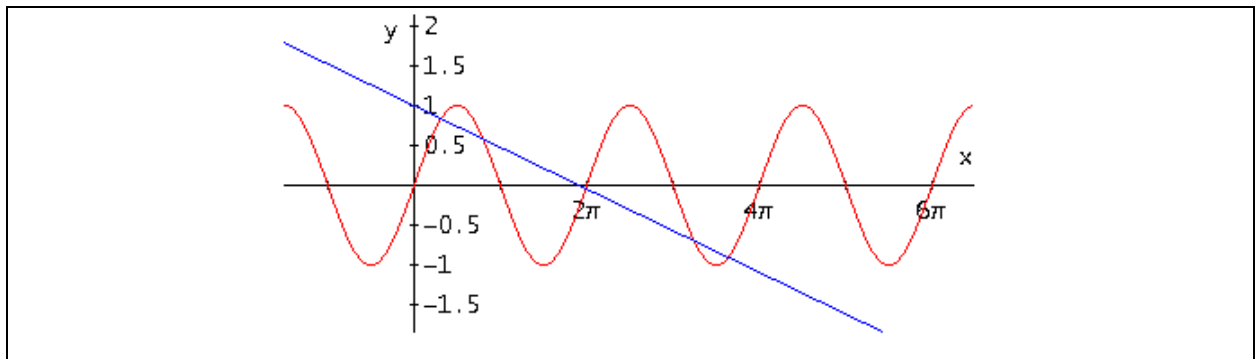
Figure 1.16

Dans *Derive*, il n'y a jamais eu de structure « RootOf » :
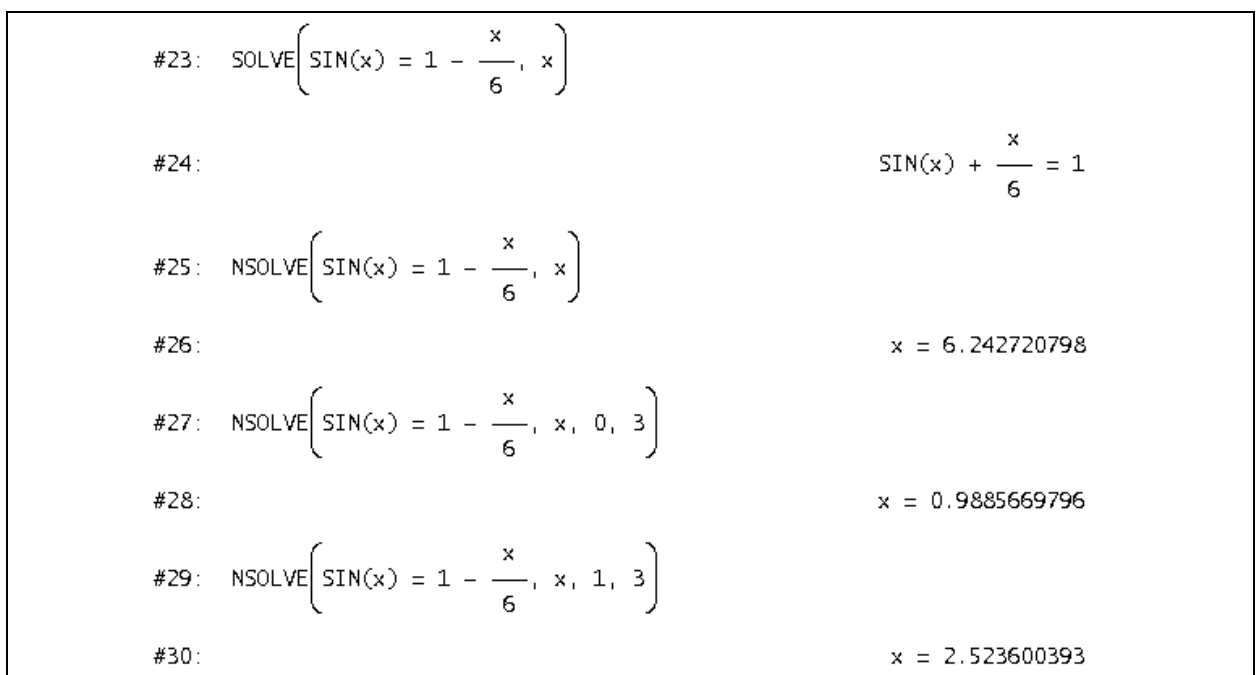
Figure 1.17

1.10 Systèmes polynomiaux   De tels systèmes sont résolubles en utilisant un algorithme (base de Gröbner) heureusement implémenté dans les systèmes symboliques.

1.11 Fonction LambertW   Voilà probablement une des belles implémentations récentes d'une fonction mathématiques (2 des auteurs étaient au symposium TIME-2004 tenu à l'ÉTS en juillet 2004. Beaucoup d'autres développeurs sont aussi venus à la conférence ACA 2009 aussi organisée à l'ÉTS en juin 2009 de même qu'à ACA 2019 en juillet 2019. Commençons par l'exemple suivant qui nous mène bien sur la piste de cette fonction spéciale!

1.11.1 Exemple On cherche toutes les solutions réelles et des solutions complexes à l'équation

$$2^x = x^6$$

Un graphique indique 2 intersections de part et d'autre de l'origine mais une troisième solution réelle (et positive) existe nécessairement puisque la fonction exponentielle 2^x finira par dominer la puissance x^6. Vu autrement, puisque cette solution est positive, on peut transformer l'équation originale en

$$\frac{\ln x}{x} = \frac{\ln 2}{6}$$

et comme la fonction

$$\frac{\ln x}{x}$$

tend vers 0 lorsque *x* tend vers l'infini, une troisième solution existe. On parvient à la localiser facilement en faisant une table de valeurs par exemple. On trouverait ceci :

```
              x    6
#34:   NSOLVE(2  = x  , x)

#35:                                        x = -0.9011325528

              x    6
#36:   NSOLVE(2  = x  , x, 0, 2)

#37:                                        x = 1.140879647

              x    6
#38:   NSOLVE(2  = x  , x, 2, ∞)

#39:                                        x = 29.21048705
```

Figure 1.18

Afin de visualiser des solutions complexes, nous appliquons la méthode introduite en 1.4.2.

La Figure 1.19 montre cela. On a pris la partie réelle et la partie imaginaire. On a ensuite fait tracer les graphes des deux courbes implicites dans une même fenêtre. On peut apercevoir, ans un voisinage de l'origine, 4 solutions complexes en plus des 2 réelles proches de 0. Ces solutions pourront être trouvées numériquement bientôt.
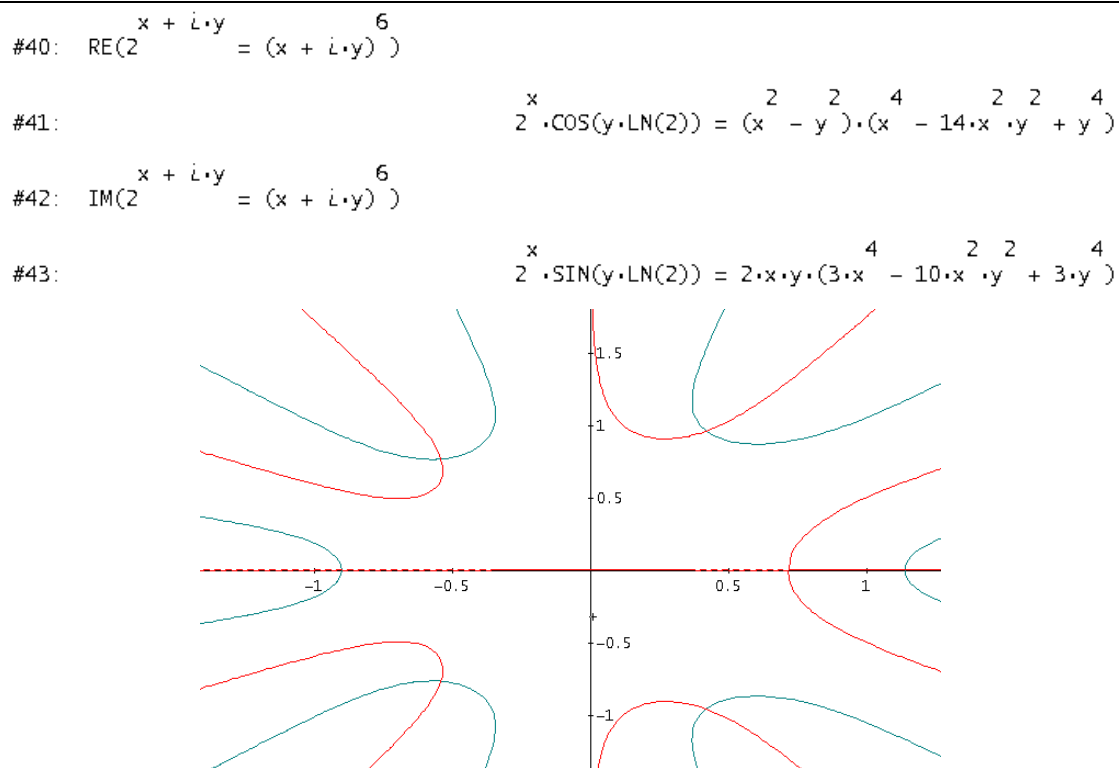
Figure 1.19

Comment expliquer ce que fait *Maple* avec la même équation? Son solveur fait appel à une fameuse fonction qui semble avoir été inventée afin de résoudre des équations du type $z\,e^{z}=w$. Afin de bien comprendre ce qui se passe, supposons que α désigne l'une des 6 racines sixièmes de 1, on doit donc résoudre l'équation $\alpha\,2^{x/6}=x$ qui est équivalente aux suivantes :

$$\alpha e^{\frac{x\ln 2}{6}}=x \Leftrightarrow xe^{\frac{-x\ln 2}{6}}=\alpha \Leftrightarrow -\frac{x\ln 2}{6}e^{\frac{-x\ln 2}{6}}=-\alpha\frac{\ln 2}{6}.$$

Donc on doit résoudre une équation du type $X\,e^{X}=Y$: dans notre exemple, $X\,e^{X}=-\dfrac{x\ln 2}{6}$ et $Y=-\dfrac{\alpha\ln 2}{6}$ avec $\alpha\in\left\{\pm 1,\pm\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}\right\}$.

Notez que la fonction $x\mapsto xe^{x}$ n'est pas bijective mais le deviendra en considérant les deux parties de part et d'autre de son minimum absolu : en inversant, on obtient deux branches dénotées par $W_0(x)$ (« branche principale ») et $W_{-1}(x)$ comme la figure 1.20 le montre. Ce sont ces deux branches seulement qui procurent des valeurs réelles (bien qu'elles donnent aussi des valeurs complexes lorsque $x<-e^{-1}$). Les autres branches W_k pour k entier donnent toujours des valeurs complexes .
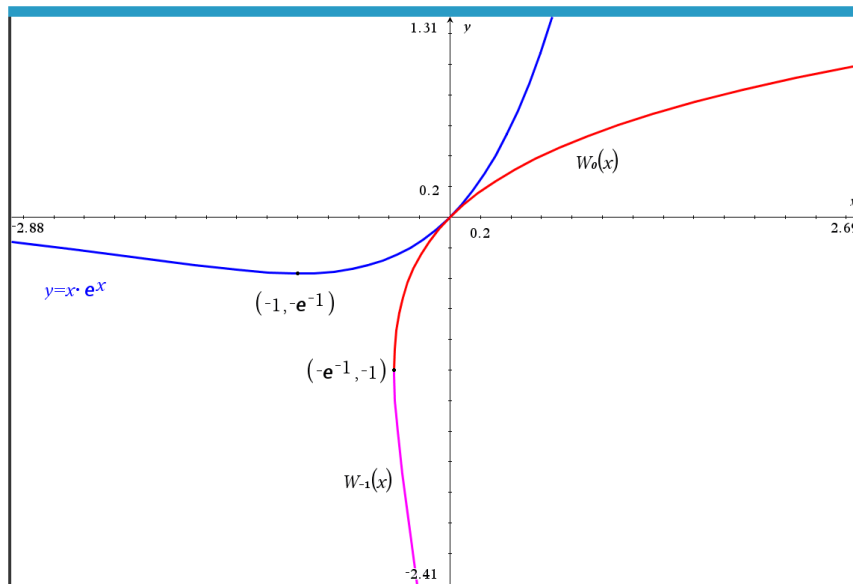
Figure 1.20

Les différentes branches de la fonction LambertW sont séparées comme suit. La courbe (donnée sous forme paramétrique ici)

$$\begin{cases}x=-t\cot t\\ y=t\end{cases},\ -\pi<t<\pi$$

à laquelle on rajoute le point $(-1,0)$ sépare la branche principale $W_0(x)$ des deux branches $W_1(x)$ et $W_{-1}(x)$. L'intervalle $]-\infty,-1]$ sépare les branches $W_1(x)$ et $W_{-1}(x)$. Finalement les autres branches sont séparées par les courbes

$$\begin{cases}x=-t\cot t\\ y=t\end{cases},\ 2k\pi<\pm t<(2k+1)\pi,\quad k=1,2,3,\cdots$$

La fonction LambertW est donc « multiforme » est c'est donc la fonction réciproque de $y=xe^{x}$. Nous pouvons maintenant poursuivre notre exemple.

Les solutions de l'équation $2^x = x^6$ sont donc $x = -\dfrac{6}{\ln 2} \text{LambertW}\left(k, -\dfrac{\alpha \ln 2}{6}\right)$. *Maple* en donne :

> $solve(2^x = x^6, x)$;

$$-\frac{6\,\text{LambertW}\left(\dfrac{1}{6}\ln(2)\right)}{\ln(2)}, \quad -\frac{6\,\text{LambertW}\left(-\dfrac{1}{6}\ln(2)\right)}{\ln(2)},$$

$$-\frac{6\,\text{LambertW}\left(-1,\, -\dfrac{1}{6}\ln(2)\right)}{\ln(2)},$$

$$-\frac{6\,\text{LambertW}\left(-\dfrac{1}{6}\ln(2)\left(\dfrac{1}{2} - \dfrac{1}{2}\,I\sqrt{3}\right)\right)}{\ln(2)},$$

$$-\frac{6\,\text{LambertW}\left(-\dfrac{1}{6}\ln(2)\left(\dfrac{1}{2} + \dfrac{1}{2}\,I\sqrt{3}\right)\right)}{\ln(2)},$$

$$-\frac{6\,\text{LambertW}\left(-\dfrac{1}{6}\ln(2)\left(-\dfrac{1}{2} - \dfrac{1}{2}\,I\sqrt{3}\right)\right)}{\ln(2)},$$

$$-\frac{6\,\text{LambertW}\left(-\dfrac{1}{6}\ln(2)\left(-\dfrac{1}{2} + \dfrac{1}{2}\,I\sqrt{3}\right)\right)}{\ln(2)}$$

$$W(z)e^{W(z)} = z \qquad y e^y = z \Leftrightarrow y = W_k(z)$$

http://www.orcca.on.ca/LambertW/

Figure 1.21

Lorsqu'on programme cette fonction avec ses différentes branches, on peut ensuite l'évaluer : nous l'avons fait dans la librairie kit_ets_mb. On trouve bien comme *Maple* ou *Mathematica* via sa fonction « ProductLog » :

ra_0$(\alpha) := \dfrac{-6}{\ln(2)} \cdot$ **kit_ets_mb\lambertw**$\left(0, \dfrac{-\alpha \cdot \ln(2)}{6}\right)$ ▸ Terminé

$\left\{ \textbf{ra\_0}(1), \textbf{ra\_0}(-1), \textbf{ra\_0}\left(\dfrac{1}{2} + \dfrac{i \cdot \sqrt{3}}{2}\right), \textbf{ra\_0}\left(\dfrac{1}{2} - \dfrac{i \cdot \sqrt{3}}{2}\right), \textbf{ra\_0}\left(\dfrac{-1}{2} + \dfrac{i \cdot \sqrt{3}}{2}\right), \textbf{ra\_0}\left(\dfrac{-1}{2} - \dfrac{i \cdot \sqrt{3}}{2}\right) \right\}$

▸ $\{ 1.14088, -0.901133, 0.420866 + 0.961768 \cdot i, 0.420866 - 0.961768 \cdot i, -0.540072 + 0.768772 \cdot i, -0.540072 - 0.768772 \cdot i \}$

ra_1$(\alpha) := \dfrac{-6}{\ln(2)} \cdot$ **kit_ets_mb\lambertw**$\left(-1, \dfrac{-\alpha \cdot \ln(2)}{6}\right)$ ▸ *Terminé*

ra_1(1) ▸ 29.2105

Figure 1.22

On peut même rajouter plusieurs autres solutions complexes (qu'on verrait d'ailleurs en agrandissant la fenêtre que donnait *Derive* à la Figure 1.19) :

$$\left\{ \mathbf{ra\_1}(\text{-}1), \mathbf{ra\_1}(1), \mathbf{ra\_1}\left(\frac{1}{2}+\frac{i \cdot \sqrt{3}}{2}\right), \mathbf{ra\_1}\left(\frac{1}{2}-\frac{i \cdot \sqrt{3}}{2}\right), \mathbf{ra\_1}\left(\frac{\text{-}1}{2}+\frac{i \cdot \sqrt{3}}{2}\right), \mathbf{ra\_1}\left(\frac{\text{-}1}{2}-\frac{i \cdot \sqrt{3}}{2}\right) \right\}$$

▸ $\{33.4659+34.0702 \cdot i, 29.2105, 30.1715-12.4531 \cdot i, 30.1715+12.4531 \cdot i, 34.8737+44.0603 \cdot i, 31.8633+23.6579 \cdot i\}$

$$\mathbf{ra\_2}(\alpha) := \frac{\text{-}6}{\ln(2)} \cdot \mathbf{kit\_ets\_mb \backslash lambertw}\left(2, \frac{\text{-}\alpha \cdot \ln(2)}{6}\right) \quad \blacktriangleright \textit{Terminé}$$

$$\left\{ \mathbf{ra\_2}(\text{-}1), \mathbf{ra\_1}(1), \mathbf{ra\_2}\left(\frac{1}{2}+\frac{i \cdot \sqrt{3}}{2}\right), \mathbf{ra\_2}\left(\frac{1}{2}-\frac{i \cdot \sqrt{3}}{2}\right), \mathbf{ra\_2}\left(\frac{\text{-}1}{2}+\frac{i \cdot \sqrt{3}}{2}\right), \mathbf{ra\_2}\left(\frac{\text{-}1}{2}-\frac{i \cdot \sqrt{3}}{2}\right) \right\}$$

▸ $\{39.8563-91.628 \cdot i, 29.2105, 42.4937-128.677 \cdot i, 41.2725-110.207 \cdot i, 39.0541-82.2787 \cdot i, 40.5925-100.934 \cdot i\}$

Figure 1.23

1.11.2 Simplifications avec LambertW   Si l'on s'en tient à la résolution d'équations, alors toute équation ramenable à la forme $y e^y = z$ dans laquelle z est un nombre donné (possiblement complexe) et dans laquelle y est l'inconnue à trouver, pourra être résolue en prenant la fonction LambertW de chaque côté de cette équation. En d'autres termes, utilisant « W » pour désigner une quelconque branche d'indice « k » de la fonction LambertW(k, z), on a donc

$$W\left(y e^y\right) = y.$$

En particulier on a (ce qui se remarquait d'ailleurs de la figure 1.21)

$$e^{W(z)} = \frac{z}{W(z)}.$$

1.11.3 Exemple   Il est facile de trouver à l'œil les solutions 2 et 4 de l'équation $x^2 = 2^x$. La troisième solution réelle (ainsi que 2 et 4), de même que toutes les solutions complexes sont faciles à trouver via LambertW. En effet, l'équation est équivalente à $x = \pm e^{x \ln(2)/2}$ qui est à son tour équivalente à

$$-x \frac{\ln 2}{2} e^{-x \ln(2)/2} = \pm \frac{\ln 2}{2}.$$

Mais alors $-x = -\dfrac{2}{\ln(2)} W\left(\pm \dfrac{\ln 2}{2}\right)$ où « W » désigne une branche de LambertW. Notons que $-\ln(2)/2 > -1/e$ et donc les branches $k = 0$ et $k = -1$ vont être utilisées pour évaluer la valeur de $W\left(-\dfrac{\ln 2}{2}\right)$ et la branche $k = 0$ sera utilisée aussi pour la valeur de $W\left(\dfrac{\ln 2}{2}\right)$. Les solutions réelles se retrouvent donc parmi les 3 valeurs suivantes :

$$-\frac{2}{\ln 2} W\left(0, \frac{\ln 2}{2}\right), \quad -\frac{2}{\ln 2} W\left(0, \frac{-\ln 2}{2}\right) \text{ et } -\frac{2}{\ln 2} W\left(-1, \frac{-\ln 2}{2}\right).$$

La première réponse donne $-0.766665\ldots$ Notons maintenant ceci : dans $W\left(y e^y\right) = y$, remplaçons y par $-\ln(x)$ et utilisant le fait que pour x non nul on a toujours $\exp(\ln(x)) = x$ et donc $\exp(-\ln(x)) = 1/x$, on trouve que $W\left(-\dfrac{\ln x}{x}\right) = -\ln x$. Mais alors l'image de la branche $k = 0$ consistant en les réels ≥ -1 et puisque $-\ln 2 > -1$, on peut écrire

$$-\frac{2}{\ln 2} W\left(0, \frac{-\ln 2}{2}\right) = -\frac{2}{\ln 2} \cdot (-\ln 2) = 2$$

Et comme l'image de la branche $k = -1$ consiste en les réels ≤ -1 et que $-\ln 4 < -1$, on peut écrire

$$-\frac{2}{\ln 2}W\left(-1, \frac{-\ln 2}{2}\right) = -\frac{2}{\ln 2}W\left(-1, \frac{-2\ln 2}{4}\right) = -\frac{2}{\ln 2}W\left(-1, \frac{-\ln 4}{4}\right) = -\frac{2}{\ln 2}\cdot(-\ln 4) = 4.$$

1.11.4 Exemple Dans $W\left(y\,e^y\right) = y$, si l'on remplace maintenant y par $\ln(x)$ et utilise encore le fait que pour x non nul on a toujours $\exp(\ln(x)) = x$, on trouve que $W(x\ln x) = \ln x$. Cela permet de montrer que la solution (positive) à l'équation $x^{x^3} = 3$ est $\sqrt[3]{3}$. En effet, un calcul montre qu'on aboutit à $x^3 \ln x = \ln 3$,, d'où $x^3 \ln(x^3) = 3\ln 3$, d'où $W(x^3 \ln(x^3)) = W(3\ln 3) = \ln 3$ et ainsi $\ln\left(x^3\right) = \ln 3$ et $x = \sqrt[3]{3}$.

1.12 Remarque et définition Dans le cas des solutions données numériquement, quelles méthodes « magiques » les systèmes symboliques utilisent-t-ils? Probablement, un « mélange » de différentes méthodes.

Nous allons présenter deux d'entre elles maintenant. La première est celle du point fixe et la seconde est celle de Newton. Chacune de ces méthodes sera étendue aux systèmes d'équations et l'utilisation d'un calculateur s'avère fort utile, voire nécessaire ici.

On dit qu'une fonction g possède un <u>point fixe</u> s'il existe un nombre r tel que $g(r) = r$. Si, de plus, la fonction g est dérivable, nous dirons que ce point fixe est un <u>attracteur</u> si $|g'(r)| < 1$.

Ainsi, on trouve facilement que les points fixes de la fonction

$$g(x) = \frac{x\left(4 + x^2\right)}{1 + 4x^2}$$

sont 0, 1 et −1 et que 0 n'est pas un attracteur. On peut alors se demander si, en itérant $g(x)$, cela va converger. Ainsi, en partant « proche » de 0, disons 0.2, nous trouverions que ça converge mais pas vers 0 mais plutôt vers 1 : la fonction « iterates » de *Derive* possède un dernier argument facultatif. La précision est limitée à 14 chiffres dans Nspire (on voit à la figure 1.25 l'affichage à « float 6 ») :

ITERATES avec *Derive :*

```
              2
          x·(4 + x )
#1:  g(x) := ─────────
              2
          1 + 4·x

#2:  ITERATES(g(x), x, 0.2, 10)

#3:  [0.2, 0.6965517241, 1.062374033, 0.9880277022, 1.002411363, 0.9995184273, 1.000096342, 0.9999807326, 1.000003853,
      0.9999992292, 1.000000154]

#4:  ITERATES(g(x), x, 0.2)

#5:  [0.2, 0.6965517241, 1.062374033, 0.9880277022, 1.002411363, 0.9995184273, 1.000096342, 0.9999807326, 1.000003853,
      0.9999992292, 1.000000154, 1, 1]
```

Figure 1.24

iterates avec TI-NspireCAS (contenu dans libraire *kit_ets_mb*):

$$g(x) := \frac{x \cdot (4+x^2)}{1+4 \cdot x^2} \qquad \text{Terminé}$$

$$kit\_ets\_mb \backslash iterates(g(x), x, 0.2, 10)$$

$$[0.2 \quad 0.696552 \quad 1.06237 \quad 0.988028 \quad 1.00241 \quad 0.999518 \quad 1.0001 \quad 0.999981 \quad 1. \quad 0.999999 \quad 1.]$$
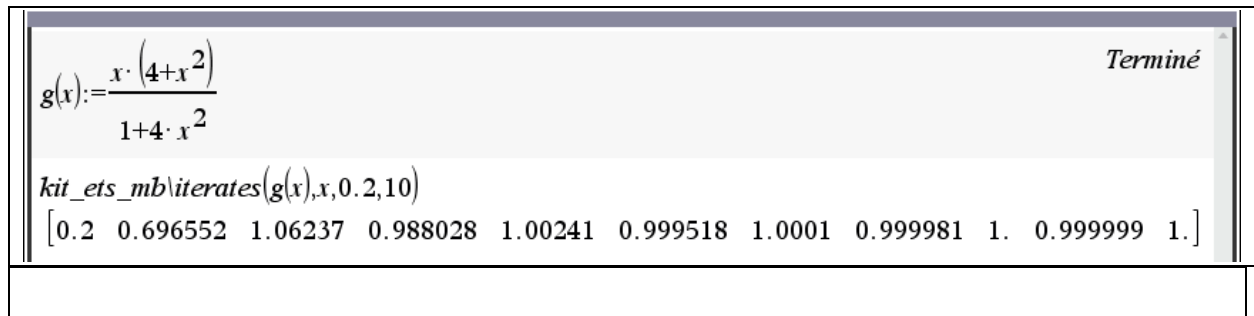
Figure 1.25

to be continued

Caused by the well-known circumstances the 26th International Conference on Applications of Computer Algebra was held virtual online.

Michel Beaudin and Michael Wester were among the organizers of the "Education session". Michael sent a link for downloading videos of the session presentations. I am very grateful that they gave permission to publish this link in our newsletter:

https://math.unm.edu/~aca/ACA/2021/education.html

Talks

1. A CAS-DGS assisted exploration of Spiric curves and their Hessians
   (Thierry Dana-Picard, Israel)
2. CAS Tools for teaching function discontinuities
   (David G. Zeitoun, Israel)
3. Automated exploration of envelopes and offsets with networking of technologies
   (Thierry Dana-Picard and Zoltán Kovács, Israel and Austria)
4. An Automated Symbolic Package to Enhance Higher Order Thinking Skills (HOTS): Critical Thinking
   (Yuzita Yaacob and Khairina Atika Mohd. Zawawi, Malaysia)
5. From hidden invariants to multiple solutions using computer algebra tools: two activities for pre-service teachers
   (I. Sinitsky and M. Sinitsky, Israel
6. Can I bring my calculator to the exam? Some reflections on the abstraction level of CAS
   (Eugenio Roanes-Lozano, Spain)
7. Using CAS in the classroom: personal thoughts (Part I)
   (Michel Beaudin, Canada)
8. Is computer algebra ready for conjecturing and proving geometric inequalities in the classroom?
   (Zoltán Kovács, Tomás Recio, Róbert Vajda, M. Pilar Vélez, Austria, Spain and Hungary)
   [Presentation]
9. Challenges and opportunities in remote teaching, learning and assessment of undergraduate mathematics
   (Elena Varbanova and Magdalina Uzunova, Bulgaria)
10. Undergraduate Mathematics: a journey from a face-to-face to a remote teaching, learning and assessment: Discussion
    (Elena Varbanova, Bulgaria)

You will find the videos on the bottom of the website under "Recordings".

Spielkarten gut gemischt / A Deck of Cards - Well Shuffled

Heinrich Ludwig, Germany

Between two rounds of a card game, the cards must be reshuffled. A commonly used method is to hold the deck of cards in one hand and use the thumb of the other hand to pull off a top portion of the deck and drop it into the other hand. You continue this until there are no more cards in the discarding hand. You could just as easily split the pile into a few blocks and reassemble the blocks into a new pile in reverse order. This type of mixing will be referred to as "rearrange-ment" in the following. Since mixing is still insufficient after a single rearrangement, the process is re-peated a few times. How good is this method? This question will be examined below using the French deck as an example.

Vector *b* represents a stack of the French deck (K = Diamonds, H = Hearts, P = Spades, T = Clubs, B = Jack, D = Queen, K = King, e.g. KB = Jack of diamonds).

```
#1:    InputMode ≔ Word

#2:    b ≔ [K2, K3, K4, K5, K6, K7, K8, K9, K10, KB, KD, KK, KA, H2, H3, H4, H5,
          H6, H7, H8, H9, H10, HB, HD, HK, HA, P2, P3, P4, P5, P6, P7, P8, P9,
          P10, PB, PD, PK, PA, T2, T3, T4, T5, T6, T7, T8, T9, T10, TB, TD, TK, TA]
```

In this stack, the order of the cards is arbitrary. If the stack is intermixed, you can only judge the degree of intermixing of the stack if you know the initial order. Therefore, it is convenient to use cards which are identified by numbers and therefore have a natural sequence. The advantage will be seen later when formulating functions that evaluate mixing. In the following, we therefore use the stack *a*, which is as-sumed to always be sorted in ascending order before shuffling.

```
#3:    a ≔ [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,
          20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37,
          38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]
```

Function **reshift(v, p)** is modeled according the manual procedure. It separates the card stack *v* into several blocks and reassembles these blocks in reverse order to form a new stack. **reshift** accepts for *v* arbitrary non-empty vectors of objects (e.g. also the French deck *b*). In the following applications, however, stack *a* with 52 number cards is used for *v*. The separators in the stack *v* are determined as follows: An index *i* runs through stack *v*. For each *i*, a Boolean variable is randomly set to 1 or 0. The probability for 1 can be specified with the parameter *p*. If the variable is equal to 1, then *v* is separated after the ith digit. The larger *p*, the more finely *v* is divided into blocks (on average when the function is repeated frequently). To the shuffling of 52 cards by hand correspond *p*-values between 0.06 and 0.13.

```
reshift(v, p, w := [], i, j) :=
  Prog
    i := DIM(v)
    j := i
    Loop
      If i < 2 exit
#5:   If RANDOM(100000) < p·100000
        Prog
          w := APPEND(w, v↓[i, ..., j])
          j := i − 1
      i :− 1
    w := APPEND(w, v↓[1, ..., j])
    RETURN w
```

When a card player shuffles, he reshuffles the deck several times. Function **shuffle(v, p, n)** takes this into account. It iterates reshift n times, where the parameters v and p correspond to those of reshuffle.

```
#7:   shuffle(v, p, n) := ITERATE(reshift(tmp, p), tmp, v, n)
```

Now the task is to evaluate the mixing process. Many criteria can be used, but the following seems particularly suitable for mixing by rearrangement. If you reshuffle a deck just a few times, you will notice that quite a few cards have a successor card which they had before the shuffle. After good shuffling, however, this should occur only once on average with 52 cards (see below). The number k of unseparated card pairs is therefore used to evaluate a shuffle.

This number k determines the function **kvalue(v)** with respect to the stack of cards v. Let $[v_1, v_2, \ldots, v_n]$ be a stack of cards. Then **kvalue** for each card v_i is the difference to its successor card v_{i+1}. If this is equal to 1, then the card has kept its successor. **kvalue** counts how many times this occurs in v. (As can be seen, **kvalue** must assume that the shuffle operation was applied to map sheet a). Because it seems mathematically "nicer" to me, the last card with number 52 should also have a successor card, namely the one with number 1. The succession is therefore to be understood cyclically.

Because mixing is a random process, k has a certain probability distribution. It depends on the type of mixing. At first, only their expected value <k> shall be considered. It is determined by the function **kmean(n, p)**. It shuffles the deck of cards a 1000 times. Parameters n and p have the same meaning as in the shuffle function.

$$\#10: \quad \text{kvalue}(v) := \left(\sum_{i=1}^{\text{DIM}(v) - 1} \chi\left(\frac{1}{2}, \text{MOD}(v_{i+1} - v_i, \text{DIM}(v)), \frac{3}{2}\right) \right) + \chi\left(\frac{1}{2}, \text{MOD}(v_1 - v_{\text{DIM}(v)}, \text{DIM}(v)), \frac{3}{2}\right)$$

```
#11:  kmean(n, p) := [n, p, AVERAGE(VECTOR(kvalue(shuffle(a, p, n)), i, 1, 1000))]

#12:  kmean(80, 0.13) = [80, 0.13, 1.003]

#13:  kmean(4, 0.13) = [4, 0.13, 29.541]
```

One recognizes: The more shifts in mixing, the lower the mean value <k>. That was to be expected. It seems to be that, with increasing reshuffling, <k> approaches a limit close to 1. So, is <k> = 1 at 52 cards an indication of a process that "completely" shuffles?

Or, conversely, may we conclude that "incomplete" mixing has occurred if k is above some bound, say 4? This is to be checked. To do this, however, one would have to define what is meant by complete mixing and one would have to know what the probability distribution of k looks like in this complete mixing.

An algorithm that shuffles completely is said to be an algorithm that produces all permutations of the deck with equal probability. Function **randperm(v)** is such one, where v is the deck of cards ($v \neq []$) that is shuffled.

What value do we get for <k>, so if "really well" mixed? To do this, we replace function **shuffle** by **randpermut** in function **kmean**, which gives us **kmean2**.

(The number of repetitions in **kmean2** was set to 100000 to minimize the error for the result <k>. Since **kmean2** thus requires 5.2E6 random numbers, but DERIVE's pseudorandom number generator (PRNG) has a period length of only 2.6E9, 0.2 percent of the supply of random numbers is already exhausted. A significant increase in the number of repetitions makes sense only if you use a better PRNG in **randperm**.)

```
         randperm(v, w := [], z, d) :=
           Prog
             d := DIM(v)
             Loop
               If d < 2 exit
#24:           z := RANDOM(d) + 1
               w := ADJOIN(v↓z, w)
               v := DELETE(v, z)
               d :- 1
             w := ADJOIN(v↓1, w)
             RETURN w
```

```
#25:  kmean2 := AVERAGE(VECTOR(kvalue(randperm(a)), i, 1, 100000))
```

```
#26:  kmean2 = 1.01734
```

As you can see, approximately <k> = 1 applies. This means that we know - what we will not have doubted before - that complete mixing is not impossible with sufficiently repeated reshuffling.

How close a mixing process - e.g., reshuffling 4 or 30 times - comes to complete mixing can be judged well only if the probability distribution of k for complete mixing is known. Unfortunately, I don't know any calculation method, but have to rely on approximating it by an empirically determined relative frequency distribution. This is done by the **histogramm(dat, below, above)** function. It takes a vector **dat** containing a great many k outcomes, and for each k value between a minimum **below** and a maximum **above**, counts how many times it is contained in **dat**. VECTOR(kvalue(randperm(a)),i,1,200000) produces a database containing 200000 k-values. The result is presented in expression #30.

```
         histogramm(dat, unt, ob, h, q, d, j, i := 1) :=
           Prog
             d := DIM(dat)
             h := VECTOR(0, j, unt, ob)
             q := VECTOR(j, j, unt, ob)
#27:         Loop
               If i > d exit
               h↓(dat↓i - unt + 1) :+ 1
               i :+ 1
             RETURN [q, h/d]'
```

```
#28:  histogramm(VECTOR(kvalue(randperm(a)), i, 1, 200000), 0, 10)
```

#30:

$$\begin{bmatrix} 0 & 0.36013 \\ 1 & 0.36752 \\ 2 & 0.188125 \\ 3 & 0.063445 \\ 4 & 0.016645 \\ 5 & 0.003485 \\ 6 & 0.00053 \\ 7 & 0.000105 \\ 8 & 10^{-5} \\ 9 & 0 \\ 10 & 5 \cdot 10^{-6} \end{bmatrix}$$

Expression #30 can be plotted in a 2D graphics window without further processing. We don't get a nice bar chart, but we do get a good impression of the relative frequency distribution of k after a random permutation.
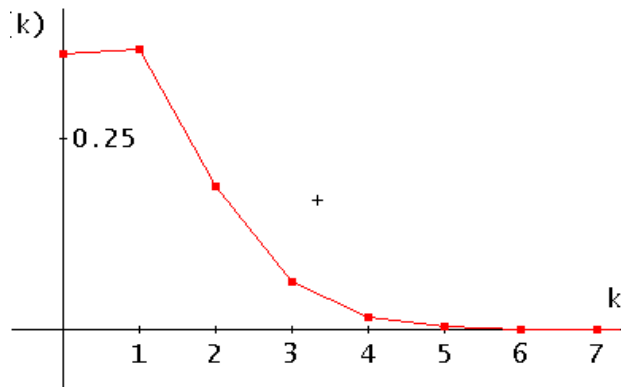
#32: histogramm(VECTOR(kvalue(shuffle(a, 0.13, 10)), i, 1, 25000), 0, 20)

#33:

0	1	2	3	4	5	6	7	8	9
0	0	$4 \cdot 10^{-5}$	0.0002	0.00088	0.00288	0.00696	0.01592	0.031	0.05128

10	11	12	13	14	15	16	17	18
0.0766	0.09828	0.1218	0.1232	0.121	0.107	0.0866	0.0616	0.03896

19	20
0.02632	0.01484

#35: `histogramm(VECTOR(kvalue(shuffle(a, 0.13, 20)), i, 1, 25000), 0, 20)`

#36:

0	1	2	3	4	5	6	7
0.01544	0.06752	0.1396	0.19312	0.19864	0.16244	0.10944	0.06208

8	9	10	11	12	13	14	15	16	17	18
0.0304	0.01416	0.00528	0.0016	0.0002	$4 \cdot 10^{-5}$	0	$4 \cdot 10^{-5}$	0	0	0

19	20
0	0

#38: `histogramm(VECTOR(kvalue(shuffle(a, 0.13, 30)), i, 1, 25000), 0, 20)`

#39:

0	1	2	3	4	5	6	7
0.16948	0.30224	0.26612	0.15916	0.06832	0.0246	0.0076	0.00124

8	9	10	11	12	13	14	15	16	17	18	19	20
0.00104	0.0002	0	0	0	0	0	0	0	0	0	0	0
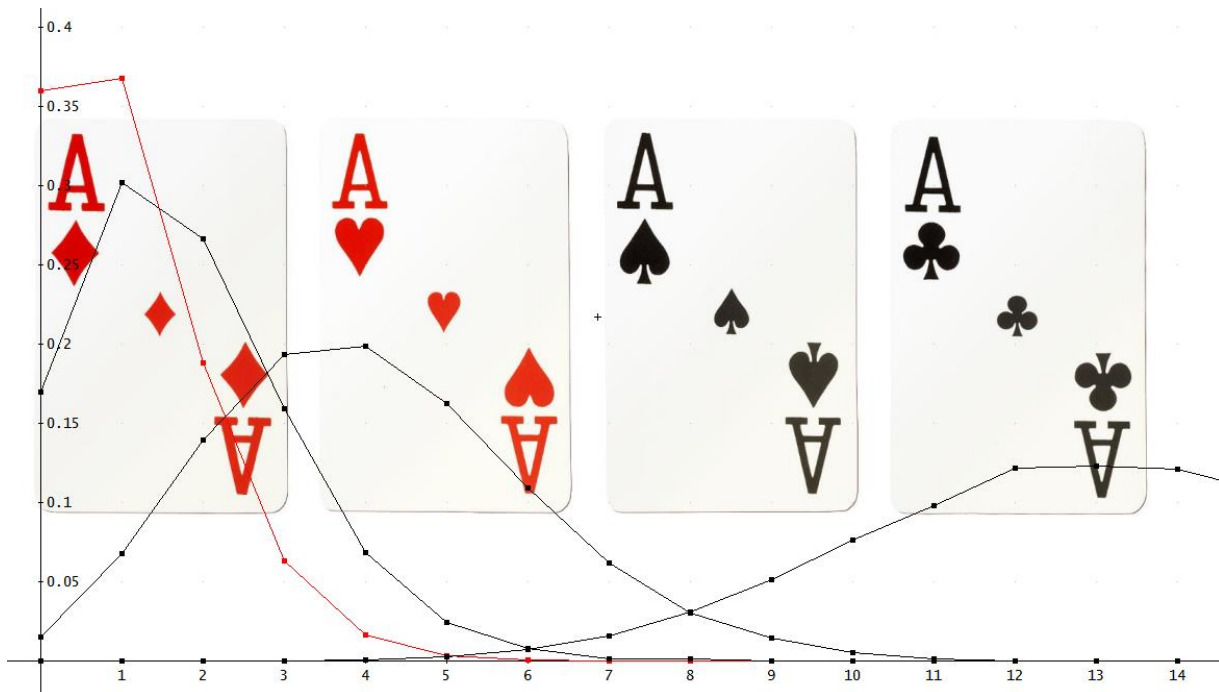
The three black graphs with maxima at 13, 4, and 1, respectively, represent the *k* distributions for 10-, 20-, and 30-fold reshuffling (RS), respectively. The graph of the distribution after a random permutation (RP) is juxtaposed with these in red.

The distinction of the three distributions after repeated shuffling (RS) from the RP distribution is, of course, only so clear because each was shuffled 25000 times. On a game night, however, shuffling is roughly estimated to be only 100 times. If one were to do a Chi^2 test on whether a RS30 sample of size 100 is consistent with the RP distribution, the result would most likely be "yes." So if card players demand "best" shuffled cards, then 30 times of shuffling is quite enough. It is completely different with

10 times shuffling. Without applying a test, it can already be seen from the first sample, the first mixing, that the quality of the RP mixing has not been achieved.

Remark:

The *k* distributions were obtained for permutations of 52 cards; they will differ for a different number of cards. Getting other distributions does not require much effort, because the functions of this dfw file are not fixed at 52. You only have to change a and repeat the above calculations. You will find that for 32 cards (Schafkopf, Skat, …) corresponding distributions differ only very slightly.

Result:

Just as many card players shuffle by moving a deck from one hand to another half a dozen times, you don't get a good shuffle of the deck by any means. Applying the number *k* as a criterion, at least 20 reshuffles are needed for a shuffle to achieve that the shuffle cannot be justifiably rejected by a critical player in the course of an evening. However, I don't know of any card player who would take the mathematical aspect so seriously as to shuffle the deck twenty times. And if he did, his teammates might not let him shuffle around much longer.

This is not a French Deck, but a "Double German Deck" which is mainly used in parts of Austria, Hungary, Slovakia).


```
#40:  b := VECTOR(k, k, 1, 20)

#41:  histogramm(VECTOR(kvalue(randperm(b)), i, 1, 200000), 0, 10)

#42:  histogramm(VECTOR(kvalue(shuffle(b, 0.13, 10)), i, 1, 25000), 0, 20)

#43:  histogramm(VECTOR(kvalue(shuffle(b, 0.13, 20)), i, 1, 25000), 0, 20)

#44:  histogramm(VECTOR(kvalue(shuffle(b, 0.13, 30)), i, 1, 25000), 0, 20)
```
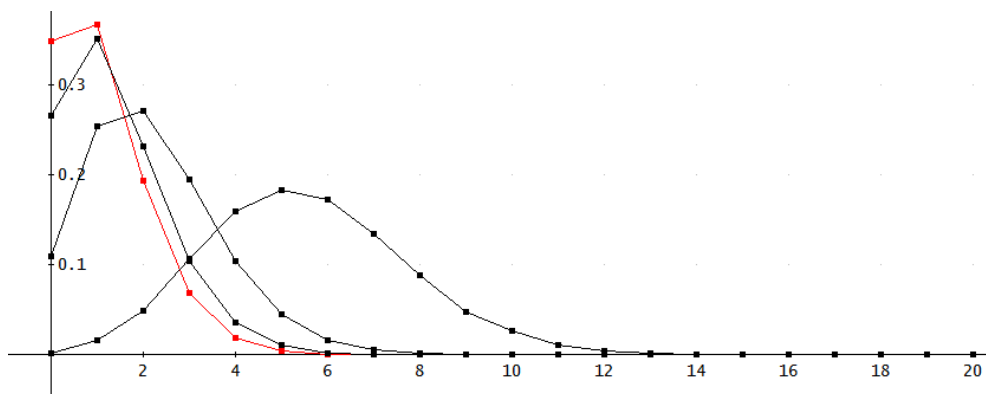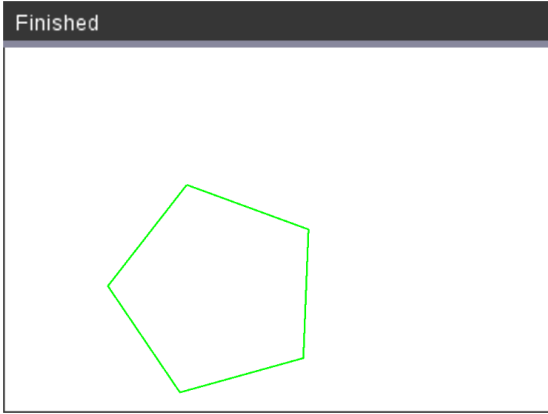

File LudwigMischen52.dfw combines Heinrich's the German original version with my English translation. This is the full German description together with all functions in German and in English as well.

In last DNL I transferred M Myers' TI-92 program to TI-Nspire and wrote, that this can only be done using the new PYTHON feature. This is wrong. It can be done without PYTHON, too as I will show below. Sorry for my wrong information. Josef


```
polygon                                                    0/17
Define polygon(va,vb,vc,vd,ve,r,g,b)=
Prgm
Local vl,vk,vl,vf,vg,vh,vi,vm,xl,yl
xl:={ }:yl:={ }
vl:=√((va−vc)²+(vb−vd)²) :vk:=vd−vb
If vk=0 Then
  If vc−va<0 Then:vm:=π
  Else:vm:=0
  EndIf
Else:vm:=sin⁻¹(vk/vl)
  If vc−va<0:vm:=π−vm
  vg:=(2·π)/ve:vh:=vc:vi:=vd
EndIf
For vf,1,ve+1
  xl:=augment(xl,{va+vl· cos(vm+vg· vf)})
  yl:=augment(yl,{vb+vl· sin(vm+vg· vf)})
EndFor
SetWindow ⁻18,18,⁻12,12:SetColor r,g,b
DrawPoly xl,yl
EndPrgm
```


Finished


```
1.1  1.2          *polygon        RAD ☐ ✕
polygon(-4,-4,2,0,6,255,0,0)              Done
polygon(-4,-4,2,0,6,100,0,100)            Done
polygon(-4,-4,2,0,5,0,255,0)              Done
polygon(0,0,7,7,9,0,0,255)                Done
```


Finished

Finished

Finished
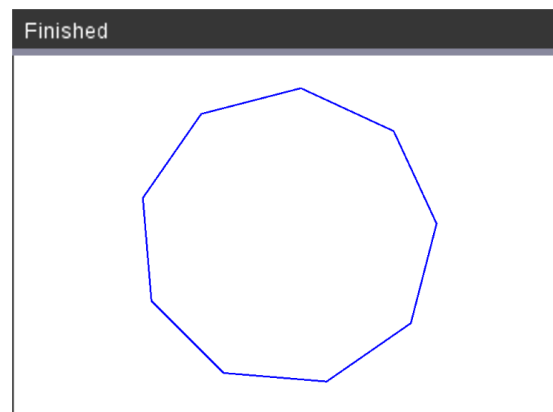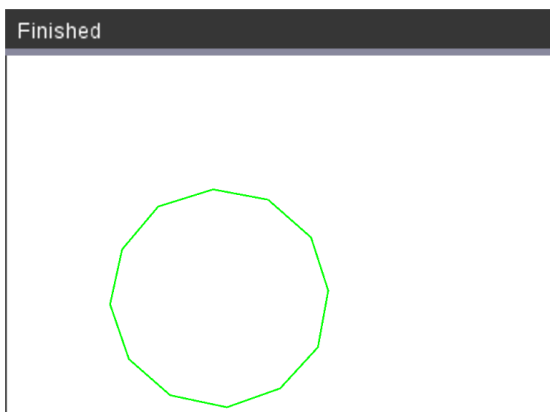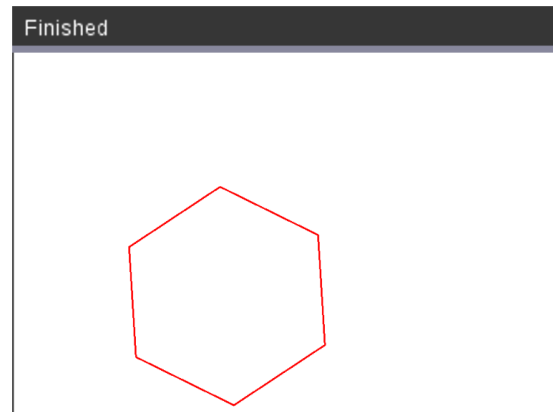
A Survey over Distributions – well known and not so well known 2

Bhuvanesh Bhatt & Josef Böhm & F. Martinez Vargas

(All TI-Nspire functions are contained in the library mathtool.tns.)

Bernoulli Distribution

In probability theory and statistics, the **Bernoulli distribution**, named after Swiss mathematician Jacob Bernoulli, is the discrete probability distribution of a random variable which takes the value 1 (= success) with probability p and the value 0 (= failed) with probability $q = 1 - p$.

$pdf2\left(statdist\left("Bernoulli",\left\{\dfrac{1}{6}\right\}\right),0\right)$	$\dfrac{5}{6}$
$pdf2\left(statdist\left("Bernoulli",\left\{\dfrac{1}{6}\right\}\right),1\right)$	$\dfrac{1}{6}$
$pdf2\left(statdist\left("Bernoulli",\left\{\dfrac{1}{6}\right\}\right),1.5\right)$	0
$cdf2\left(statdist\left("Bernoulli",\left\{\dfrac{1}{6}\right\}\right),0\right)$	$\dfrac{5}{6}$
$cdf2\left(statdist\left("Bernoulli",\left\{\dfrac{1}{6}\right\}\right),1.5\right)$	1
$cdf2\left(statdist\left("Bernoulli",\left\{\dfrac{1}{6}\right\}\right),0.5\right)$	$\dfrac{5}{6}$

```
"pdf2" erfolg. gespeichert

EndIf
If disttype="bernoulli" Then
  If xx=0:Return 1−pars[1]
  If xx=1:Return pars[1]
  Return 0
EndIf
```

```
cdf2                                    6/56
EndIf
If disttype="bernoulli" Then
  If xx<0:Return 0
  If xx<1:Return 1−pars[1]
  Return 1
EndIf
```


$f26(x)=cdf2(statdist("bernoulli",\{p\}),x)$
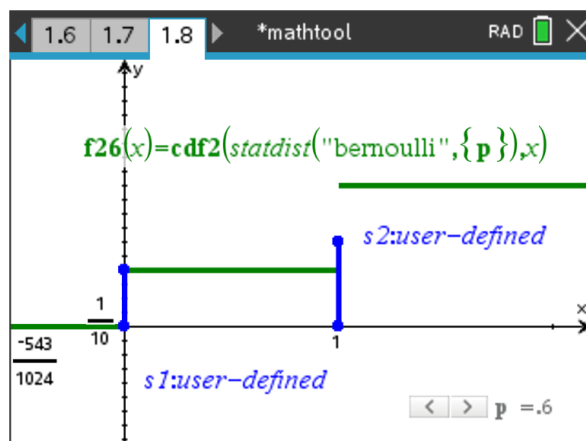
Binomial distribution

The binomial distribution is one of the most important discrete probability distributions. It is frequently used to model the number of successes in a sample of size n drawn with replacement from a population of size N. Let p the probability of a success for one draw and k the number of successes for n draws.

Density function: $f_{n,p}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0,1,2,...,n$, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Distribution function: $F_{n,p}(X \leq k) = \sum_{i=0}^{\lfloor k \rfloor} \binom{n}{i} p^i (1-p)^{n-i}$

Three examples:

1   In certain cases, kidney transplantation is successful in about 75% of patients.
    What is the probability that exactly 50% of 6 patients will achieve the desired success with this operation?
    What is the probability that at least 2 of these 6 patients will have the desired outcome?

2   Pat and Carl roll a die 50 times. Pat shall guess how often numbers 5 or 6 will be rolled. If then number of fives and sixes is equal or less Pat's guess, then Pat will win. The lower the number she estimates, the higher is her profit. What is the lowest number Pat should guess if she wants to achieve a winning probability of more than 75%?

3   Bob practices penalties for football. He knows from his experience that he is successful in 8 of 10 times. He wants to practice until he has scored 50 hits. How many penalties must he shoot that the probability to reach his goal is greater than 95%?

Answers:

1   $f_{6,\,0.75}(X=3) \sim 0.132$, $f_{6,\,0.75}(X \geq 2) = 1 - f_{6,\,0.75}(X \leq 1) = 1 - F_{6,\,0.75}(x=1) \sim 0.995$

2   $f_{50,\,1/3}(X \leq k) > 0.75$, $F_{50,\,1/3}(x=k) > 0.75$ (successes?) $k = 19$

3   $f_{n,\,0.8}(X \geq 50) > 0.95$, $1 - F_{n,\,0.8}(x \leq 49) > 0.95 \Leftrightarrow F_{n,\,0.8}(x \leq 49) < 0.05$ (sample size ?) $n = 69$

The Statistics App of TI-92 and Voyage200 support but the last question cannot be answered. (Help yourself with a table!)

Treating the problems with TI-NspireCAS:

Please note the functions *invBinom* and *invBinomN*.

Density- and distribution function are implemented in *Derive*. Inverse problems can only be solved by using tables:

```
BINOMIAL_DENSITY(3, 6, 0.75) = 0.1318359375

1 - BINOMIAL_DISTRIBUTION(1, 6, 0.75) = 0.9953613281
```

$$\text{NSOLVE}\left(\text{BINOMIAL\_DISTRIBUTION}\left(k, 50, \frac{1}{3}\right) = 0.75, k\right)$$

does not work!

$$\text{TABLE}\left(\text{BINOMIAL\_DISTRIBUTION}\left(k, 50, \frac{1}{3}\right), k, 15, 20\right)$$

$$\begin{bmatrix} 15 & 0.3689669031 \\ 16 & 0.4867947092 \\ 17 & 0.6046225153 \\ 18 & 0.7126313375 \\ 19 & 0.8035861352 \\ 20 & 0.8740761034 \end{bmatrix}$$

```
TABLE(BINOMIAL_DISTRIBUTION(49, n, 0.8), n, 65, 70)
```

$$\begin{bmatrix} 65 & 0.2154135421 \\ 66 & 0.1547978494 \\ 67 & 0.1077315468 \\ 68 & 0.07269329945 \\ 69 & 0.04761329075 \\ 70 & 0.03030808475 \end{bmatrix}$$

```
BINOMIAL_DISTRIBUTION(x, 30, p)
```

$$\text{VECTOR}\left(\begin{bmatrix} x & 0 \\ x & \text{BINOMIAL\_DENSITY}(x, 30, p) \end{bmatrix}, x, 0, 30\right)$$
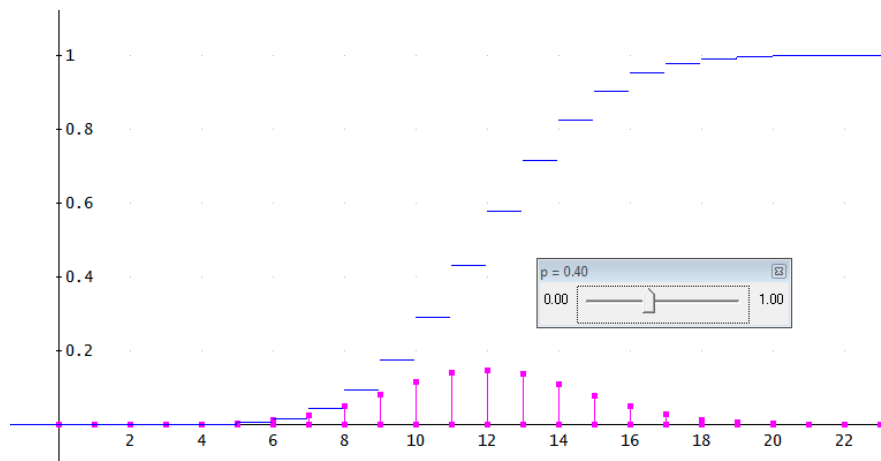
Ejemplo

En un examen formado por 20 preguntas, cada una de las cuales se responde declarando "verdadero" o "falso", el alumno sabe que, históricamente, en el 75% de los casos la respuesta correcta es "verdadero" y decide responder al examen tirando dos monedas: pone "falso" si ambas monedas muestran una cara y "verdadero" si al menos hay una cruz. Se desea saber cual es la probabilidad de que tenga más de 14 aciertos.

Hay que proporcionarle a Epidat 4 los parámetros de la distribución binomial y el punto k a partir del cual se calculará la probabilidad. En este caso n = 20, p = 0,75 y el punto k = 14. (Felipe Martinez Vargas, Mexico)

Example

In an exam consisting of 20 questions, each of which is answered by stating "true" or "false", the student knows that, historically, in 75% of the cases the correct answer is "true" and decides to answer the exam by flipping two coins: is set to "false" if both coins show heads and "true" if there is at least one tail. You want to know what is the probability that you have more than 14 hits.

The function must be provided with the parameters of the binomial distribution and the k-point from which the probability will be calculated. In this case n = 20, p = 0.75 and the point k = 14. (Translation supported by translation program DeepL.)

```
BinomialDistribution(14, 20, 0.75)
```

≪ PARAMETERS ≫	⊢————⊣
Number of Trials (n)	20
Event probability (p)	0.75
≪ PROBABILITY ≫	⊢————⊣
Number of events (k)	14
P(K = k) =	0.1686092932
P(K ≤ k) =	0.3828273378
P(K ≥ k) =	0.7857819553
P(K < k) =	0.2142180446
P(K > k) =	0.6171726621
Mean	15
Median	15
'Mode	15.75
Standard deviation	1.936491673
Variance	3.75
Skewness	−0.2581988897
Ex. kurtosis	−0.03333333333

Felipe Martinez Vargas produced a function which enables solving inverse problems for binomial distribution: ICDF_BINOM.

He developed an auxiliary function Incomplete_β, which is quite different from *DERIVE*'s INCOMPLETE_BETA function.

The example and BinomialDistribution are from his extended file PDF Y CDF.dfw.

Binomial_Martinez.dfw is the respective part treating the binomial distribution.

See below how problems 2 and 3 from above can be solved.

$$\text{ICDF\_BINOM}\left(0.75, 50, \frac{1}{3}\right) = 18.38560372$$

NSOLUTIONS(DISTR_BINOM_N(49, n, 0.8, 1) = 0.05, n)

[68.88782103]

ICDF_BINOM(pro, n, p) := RHS(NSOLVE(DISTR_BINOM_N(x, n, p, 1) = pro, x))

ICDF_BINOM(0.3828273378, 20, 0.75) = 14

Triangular Distribution

The triangular distribution is also called "Lack of knowledge Distribution". It is used as a simulation for business decision making. It is also used in connection with network techniques like CPM and PERT.
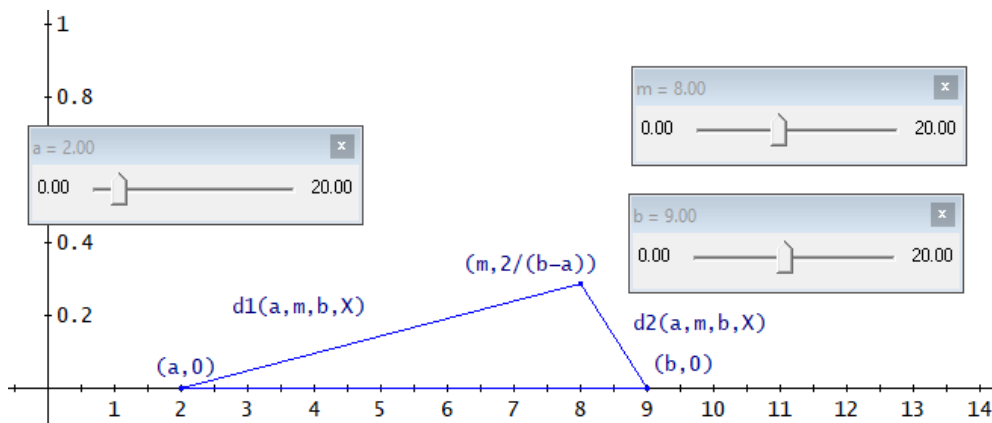
A random variable has its minimum value *a,* its maximum value *b* and its most likely value *m*. The area under the triangle must be 1, so $p(m) = 2/(b–a)$.

$$\begin{bmatrix} a & 0 \\ b & 0 \\ m & \dfrac{2}{b-a} \\ a & 0 \end{bmatrix}$$

$$\left(d1(a, m, b, x) := \dfrac{\dfrac{2}{b-a}}{m-a} \cdot (x-a) \right) = d1(a, m, b, x) := \dfrac{2 \cdot (x-a)}{(a-b) \cdot (a-m)}$$

$$\left(d2(a, m, b, x) := \dfrac{\dfrac{2}{b-a}}{m-b} \cdot (x-b) \right) = d2(a, m, b, x) := \dfrac{2 \cdot (x-b)}{(a-b) \cdot (b-m)}$$

Great occasion to introduce sliders!

The density function:

```
pdf_triangle(a, m, b, x) :=
  If ¬ a ≤ m ≤ b
     "wrong parameters"
     If x < a
        0
        If x ≤ m
           d1(a, m, b, x)
           If x ≤ b
              d2(a, m, b, x)
              0
```

The distribution function:

$$\int_{a}^{x} d1(a, m, b, u)\, du = \frac{x^2 - 2 \cdot a \cdot x + a^2}{(a - b) \cdot (a - m)}$$

$$\int_{a}^{m} d1(a, m, b, u)\, du + \int_{m}^{x} d2(a, m, b, u)\, du = \frac{x^2 - 2 \cdot b \cdot x + a \cdot (b - m) + b \cdot m}{(a - b) \cdot (b - m)}$$

```
cdf_triangle(a, m, b, x) :=
  If ¬ a ≤ m ≤ b
     "wrong parameters"
     If x < a
        0
        If x ≤ m
           ∫(d1(a, m, b, u), u, a, x)
           If x ≤ b
              ∫(d1(a, m, b, u), u, a, m) + ∫(d2(a, m, b, u), u, m, x)
              1
```
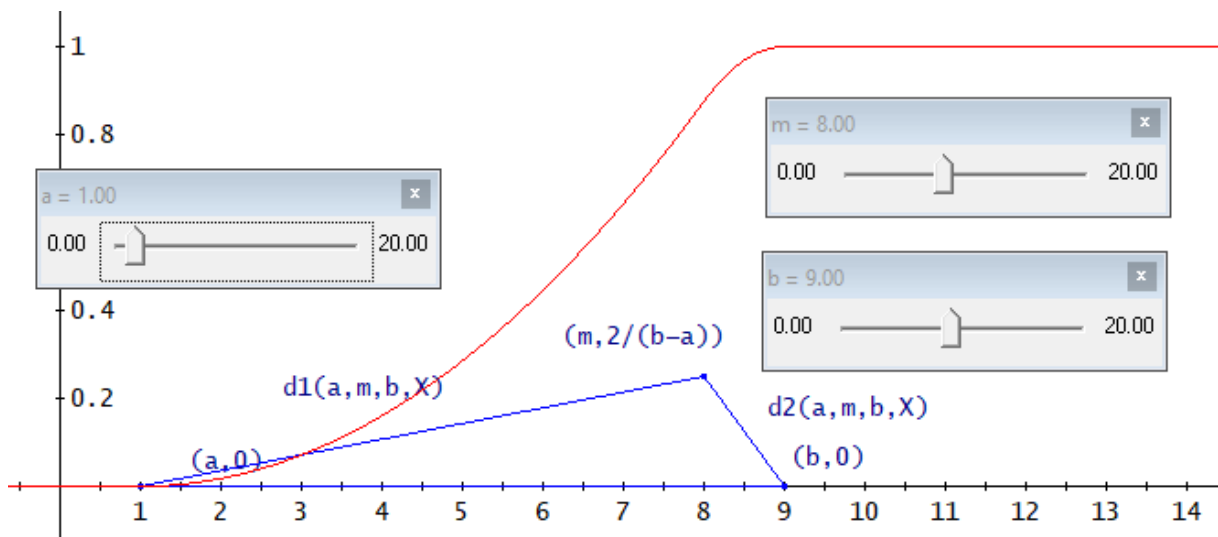

Introducing sliders enable a pretty animation.

Example

A certain product is expected to give a minimum profit 150 000, a maximum profit 300 000 and a most likely profit 240 000.

a) What's the probability that the profit will be utmost 260 000?
b) What's the probability the profit will be between 200 000 and 260 000?
c) What's the probability that the profit will be at least 250 000?
d) What's the expected value of the profit?
e) What's its variance?

Answers are given next page.

```
cdf_triangle(150, 240, 300, 260) = 0.8222222222

cdf_triangle(150, 240, 300, 260) – cdf_triangle(150, 240, 300, 200) = 0.637037037

1 – cdf_triangle(150, 240, 300, 250) = 0.2777777777
```

Expected value - Mean

$$\int_a^m x \cdot d1(a, m, b, x)\, dx + \int_m^b x \cdot d2(a, m, b, x)\, dx = \frac{a + b + m}{3}$$
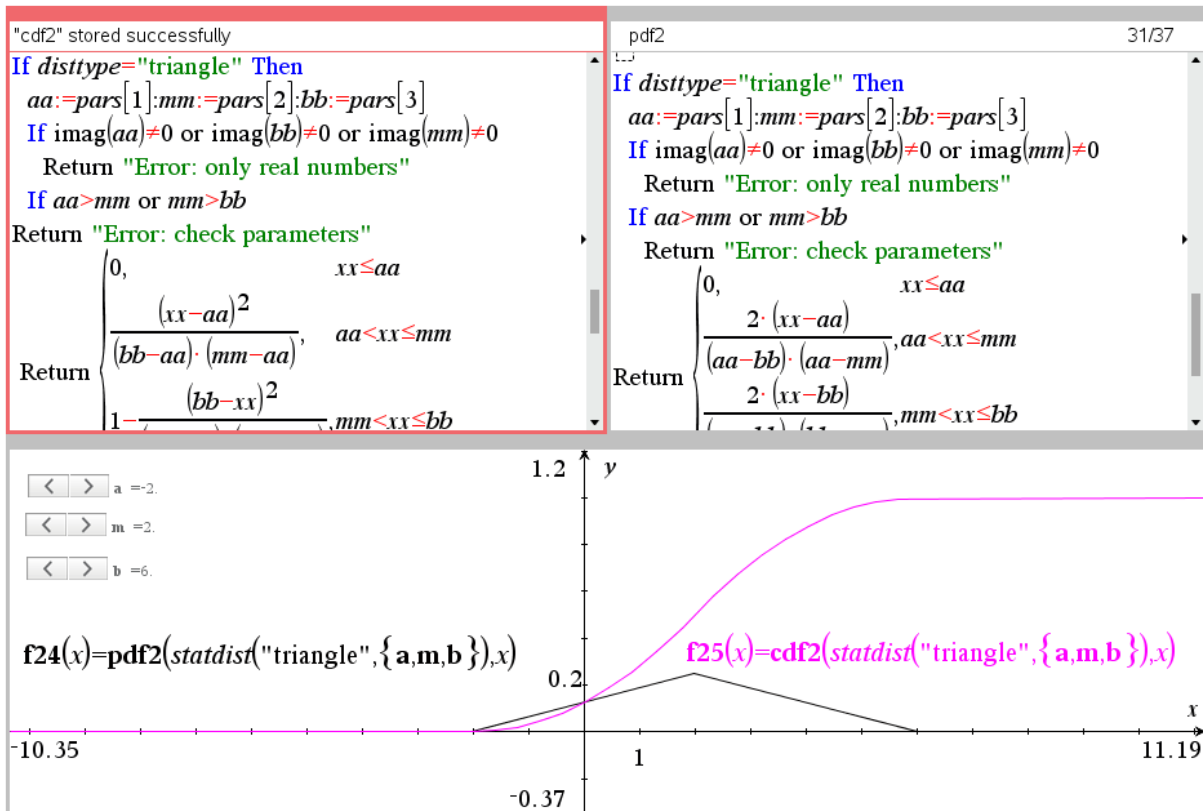
230

Variance

$$\int_a^m x^2 \cdot d1(a, m, b, x)\, dx + \int_m^b x^2 \cdot d2(a, m, b, x)\, dx - \left(\frac{a + b + m}{3}\right)^2$$

$$\frac{a^2 - a\cdot(b + m) + b^2 - b\cdot m + m^2}{18}$$

$$\frac{150^2 - 150\cdot(300 + 240) + 300^2 - 300\cdot240 + 240^2}{18}$$

950

It is no problem to transfer this to the TI-Nspire library mathtool.tns.

$mathtool\backslash cdf2(statdist("Triangle",\{150,240,300\}),260)$ $\qquad$ 0.822222222222

$mathtool\backslash cdf2(statdist("Triangle",\{150,240,300\}),260)-mathtool\backslash cdf2(statdist("Triangle",\{150,240,300\}),200$

$\qquad$ 0.637037037037

$1-mathtool\backslash cdf2(statdist("Triangle",\{150,240,300\}),250)$ $\qquad$ 0.277777777778

DelVar a,b,m $\qquad$ *Done*

⚠ $\displaystyle\int_a^m \frac{x\cdot 2\cdot(x-a)}{(a-b)\cdot(a-m)}dx + \int_m^b \frac{x\cdot 2\cdot(x-b)}{(a-b)\cdot(b-m)}dx$ $\qquad$ $\dfrac{a+b+m}{3}$

$\dfrac{a+b+m}{3}\Big|a=150 \text{ and } b=300 \text{ and } m=240$ $\qquad$ 230

⚠ $\displaystyle\int_a^m \frac{x^2\cdot 2\cdot(x-a)}{(a-b)\cdot(a-m)}dx + \int_m^b \frac{x^2\cdot 2\cdot(x-b)}{(a-b)\cdot(b-m)}dx - \left(\frac{a+b+m}{3}\right)^2$ $\qquad$ $\dfrac{a^2}{18} - \dfrac{a\cdot(b+m)}{18} + \dfrac{b^2}{18} - \dfrac{b\cdot m}{18} + \dfrac{m^2}{18}$

$\dfrac{a^2}{18} - \dfrac{a\cdot(b+m)}{18} + \dfrac{b^2}{18} - \dfrac{b\cdot m}{18} + \dfrac{m^2}{18}\Big|a=150 \text{ and } b=300 \text{ and } m=240$ $\qquad$ 950

Trapezoidal Distribution

Trapezoidal distributions seem to be appropriate for modelling the duration and the form of a phenomenon which may be represented by three stages. The first stage can be viewed as a growth-stage, the second corresponds to a relative stability and the third one represents a decline (decay).

$$\left(h\_ := \left(\text{SOLUTIONS}\left(\frac{(b-a+n-m)\cdot h}{2} = 1, h\right)\right)_1\right) = h\_ := -\frac{2}{a-b+m-n}$$

$$\begin{bmatrix} a & 0 \\ b & 0 \\ n & h\_ \\ m & h\_ \\ a & 0 \end{bmatrix}$$
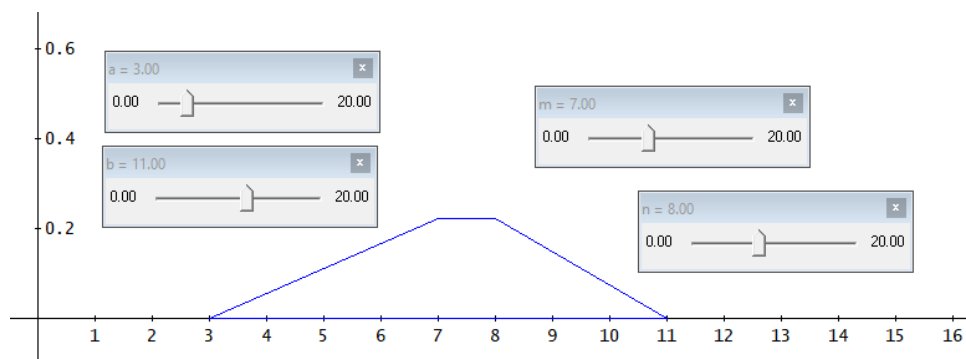
$$\left(t1(a, m, n, b, x) := \frac{h\_}{m - a} \cdot (x - a)\right) = t1(a, m, n, b, x) := \frac{2 \cdot (x - a)}{(a - b + m - n) \cdot (a - m)}$$

$$(t2(a, m, n, b, x) := h\_) = t2(a, m, n, b, x) := -\frac{2}{a - b + m - n}$$

$$\left(t3(a, m, n, b, x) := \frac{h\_}{n - b} \cdot (x - b)\right) = t3(a, m, n, b, x) := \frac{2 \cdot (x - b)}{(a - b + m - n) \cdot (b - n)}$$

```
pdf_trap(a, m, n, b, x) :=
  If ¬ a ≤ m ≤ n ≤ b
     "wrong parameters"
     If x < a
        0
        If x < m
           t1(a, m, n, b, x)
           If x < n
              t2(a, m, n, b, x)
              If x < b
                 t3(a, m, n, b, x)
                 0

pdf_trap(5, 7, 10, 12, 6.5) = 0.15
```

Let's construct the distribution function assisted by the CAS:

$$\int_{a}^{x} t1(a, m, n, b, u) \, du = \frac{x^2 - 2 \cdot a \cdot x + a^2}{(a - b + m - n) \cdot (a - m)}$$

$$\int_{a}^{m} t1(a, m, n, b, u) \, du + \int_{m}^{x} t2(a, m, n, b, u) \, du = -\frac{2 \cdot x - a - m}{a - b + m - n}$$

$$-\frac{2 \cdot n - a - m}{a - b + m - n} + \int_{n}^{x} t3(a, m, n, b, u) \, du$$

$$\frac{x^2 - 2 \cdot b \cdot x + a \cdot (b - n) + b \cdot m - n \cdot (m - n)}{(a - b + m - n) \cdot (b - n)}$$

Distribution function

```
cdf_trap(a, m, n, b, x) :=
  If ¬ a ≤ m ≤ n ≤ b
     "wrong parameters"
     If x < a
        0
        If x ≤ m
           (x^2 - 2·a·x + a^2)/((a - b + m - n)·(a - m))
           If x ≤ n
              - (2·x - a - m)/(a - b + m - n)
              If x ≤ b
                 (x^2 - 2·b·x + a·(b - n) + b·m - n·(m - n))/((a - b + m - n)·(b - n))
                 1

cdf_trap(5, 7, 10, 12, 6.5) = 0.1125
```
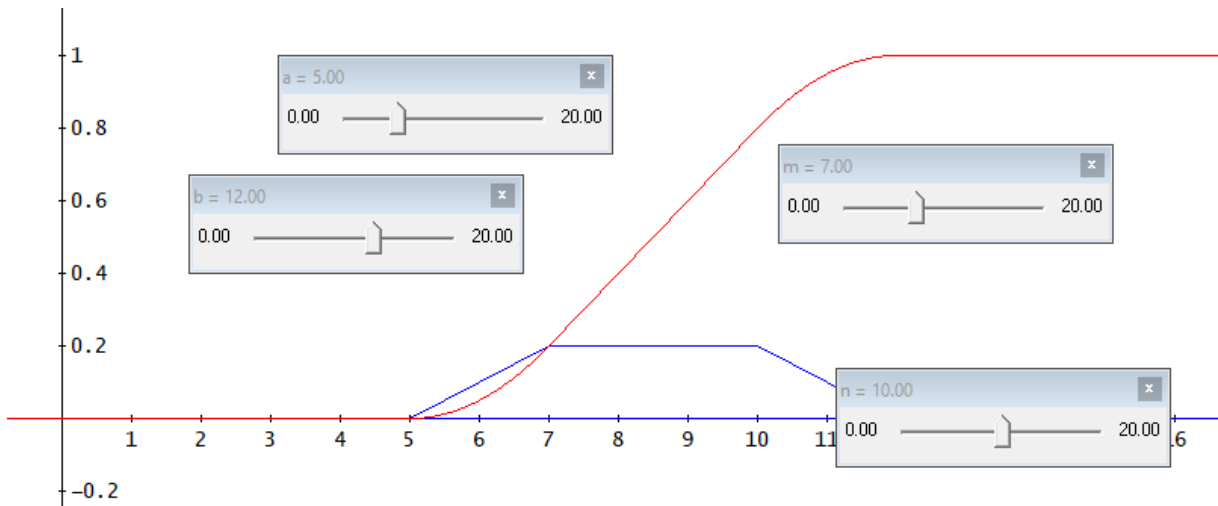
$$\frac{a+b+m}{3}$$

$$\frac{a+b+m}{3}\Big|a{=}150 \text{ and } b{=}300 \text{ and } m{=}240 \qquad 230$$

$$\int_a^m \frac{x^2\cdot 2\cdot(x-a)}{(a-b)\cdot(a-m)}\,dx + \int_m^b \frac{x^2\cdot 2\cdot(x-b)}{(a-b)\cdot(b-m)}\,dx - \left(\frac{a+b}{3}\right.$$

$$\frac{a^2}{18} - \frac{a\cdot(b+m)}{18} + \frac{b^2}{18} - \frac{b\cdot m}{18} + \frac{m^2}{18}$$

$$\frac{a^2}{18} - \frac{a\cdot(b+m)}{18} + \frac{b^2}{18} - \frac{b\cdot m}{18} + \frac{m^2}{18}\Big|a{=}150 \text{ and } b{=}300 \,\diamond$$

$$950$$

$$mathtool\backslash pdf2\big(statdist("Trapezoid",\{5,7,10,12\}),6\blacktriangleright$$
$$0.150000000000$$

$$mathtool\backslash cdf2\big(statdist("Trapezoid",\{5,7,10,12\}),6\blacktriangleright$$
$$0.112500000000$$

pdf2   35/44

If *disttype*="trapezoid" Then
  $aa:=pars[1]:mm:=pars[2]:nn:=pars[3]:bb:=pars[4]$
  If $\text{imag}(aa)\neq 0$ or $\text{imag}(bb)\neq 0$ or $\text{imag}(mm)\neq 0$ or $\text{imag}(n$
  If $aa>mm$ or $mm>nn$ or $nn>bb$:   Return "Error: chec

$$\text{Return} \begin{cases} 0, & xx\leq aa \\[2mm] \dfrac{2\cdot(xx-aa)}{(aa-bb+mm-nn)\cdot(aa-mm)}, & aa<xx\leq mm \\[3mm] \dfrac{2}{aa-bb+mm-nn}, & mm<xx\leq nn \\[2mm] \dfrac{2\cdot(xx-bb)}{} & nn<xx<bb \end{cases}$$

"cdf2" stored successfully

If *disttype*="trapezoid" Then
  $aa:=pars[1]:mm:=pars[2]:nn:=pars[3]:bb:=pars[4]$
  If $\text{imag}(aa)\neq 0$ or $\text{imag}(bb)\neq 0$ or $\text{imag}(mm)\neq 0$ or $\text{imag}(n$
  If $aa>mm$ or $mm>nn$ or $nn>bb$:Return "Error: Check p

$$\text{Return} \begin{cases} 0, \\[2mm] \dfrac{(xx-aa)^2}{(aa-bb+mm-nn)\cdot(aa-mm)}, \\[3mm] \dfrac{-(2\cdot xx-aa-mm)}{aa-bb+mm-nn}, \end{cases}$$

Valuable Links:

https://statistik.econ.kit.edu/download/WT5.pdf

https://www.statisticshowto.com/trapezoidal-distribution/

Generalized Trapezoidal Distribution

The generalized trapezoidal distribution allows more flexibility in modelling these stages. Specifically, the growth and decay stages may exhibit nonlinear convex or concave behaviour. In addition, the stability stage may be linearly increasing or decreasing.

#1: $f1(a, m, u, x) := \dfrac{u}{m - a} \cdot \left(\dfrac{x - a}{m - a}\right)^{u - 1}$

#2: $f2(m, n, k, x) := \dfrac{2}{(k + 1) \cdot (n - m)^2} \cdot ((1 - k) \cdot x + k \cdot n - m)$

#3: $f3(n, b, v, x) := \dfrac{v}{b - n} \cdot \left(\dfrac{b - x}{b - n}\right)^{v - 1}$

#4: `f1(0, m, u, x)`

#5: `f2(m, n, k, x)`

#6: `f3(n, 1, v, x)`

In case of $u = v = 2$, then growth and decay are of linear form respectively. m is the lower bound of the more or less stable phase, n is the upper one. k is the ratio $f(m)/f(n)$. $f1$, $f2$ and $f3$ are the base functions for modelling the three phases.

Sliders for all parameters encourage experimenting – but it needs "weights" to form a suitable density function (for details see the links on page 44):

Transition between the phases must be continuous and the area under function must be 1.

See below one example with only linear phases:
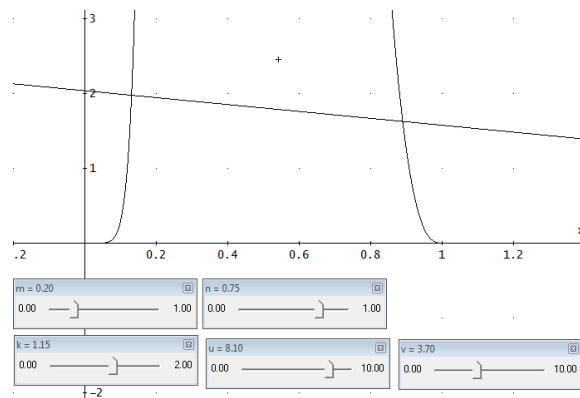
`[f1(0, 0.2, 2, x), f2(0.2, 0.8, 0.5, x), f3(0.8, 1, 2, x)]`
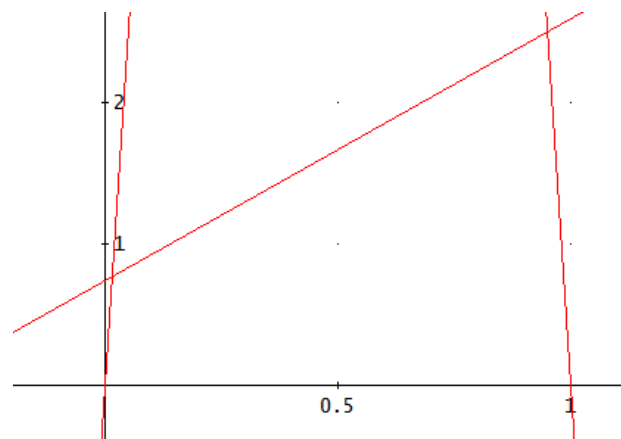
The "weight functions" are derived in [1]:

#7: $p1(a, m, n, b, k, u, v) := \dfrac{2{\cdot}k{\cdot}(m - a){\cdot}v}{2{\cdot}k{\cdot}(m - a){\cdot}v + (k + 1){\cdot}(n - m){\cdot}u{\cdot}v + 2{\cdot}(b - n){\cdot}u}$

#8: $p2(a, m, n, b, k, u, v) := \dfrac{(k + 1){\cdot}(n - m){\cdot}u{\cdot}v}{2{\cdot}k{\cdot}(m - a){\cdot}v + (k + 1){\cdot}(n - m){\cdot}u{\cdot}v + 2{\cdot}(b - n){\cdot}u}$

#9: $p3(a, m, n, b, k, u, v) := \dfrac{2{\cdot}(b - n){\cdot}u}{2{\cdot}k{\cdot}(m - a){\cdot}v + (k + 1){\cdot}(n - m){\cdot}u{\cdot}v + 2{\cdot}(b - n){\cdot}u}$

#10: `p1(a, m, n, b, k, u, v)·f1(a, m, u, x)`

#11: `p2(a, m, n, b, k, u, v)·f2(m, n, k, x)`

#12: `p3(a, m, n, b, k, u, v)·f3(n, b, v, x)`

So, #10, #11 and #12 are the "pieces" of the pdf. Next example uses the base functions from above together with their respective weights to produce the density function for $a = 0$, $b = 1$ and the "stable phase" between $m = 0.2$ and $n = 0.8$.

#14: `p1(0, 0.2, 0.8, 1, 0.5, 2, 2)·f1(0, 0.2, 2, x)`

#15: `p2(0, 0.2, 0.8, 1, 0.5, 2, 2)·f2(0.2, 0.8, 0.5, x)`

#16: `p3(0, 0.2, 0.8, 1, 0.5, 2, 2)·f3(0.8, 1, 2, x)`
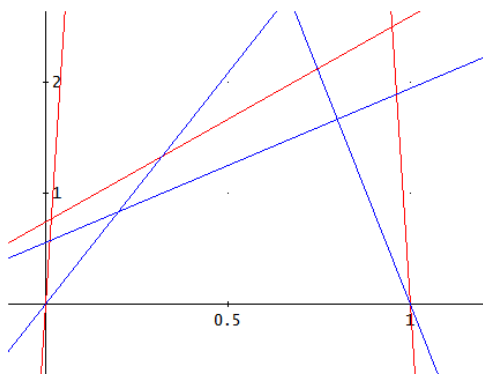
#14, #15, #16 in blue   #18 in black
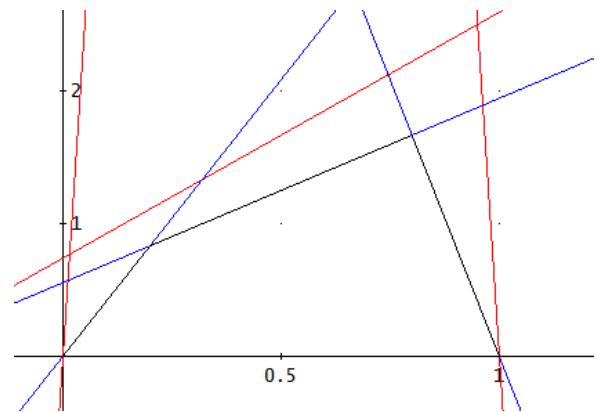
```
pdf_gen_trap(a, m, n, b, k, u, v, x) :=
  If x < a
     0
     If x < m
        p1(a, m, n, b, k, u, v)·f1(a, m, u, x)
#17:        If x < n
           p2(a, m, n, b, k, u, v)·f2(m, n, k, x)
           If x < b
              p3(a, m, n, b, k, u, v)·f3(n, b, v, x)
              0
```

#18: `pdf_gen_trap(0, 0.2, 0.8, 1, 0.5, 2, 2, x)`

#19: $\displaystyle\int_0^1$ `pdf_gen_trap(0, 0.2, 0.8, 1, 0.5, 2, 2, x) dx = 1`

Some examples:

```
pdf_gen_trap(0, 0.2, 0.6, 1, 1, 1.5, 1.5, x)

pdf_gen_trap(0, 0.2, 0.6, 1, 1.5, 2.5, 2.5, x)

pdf_gen_trap(0, 0.2, 0.6, 1, 1.2, 0.5, 0.8, x)

pdf_gen_trap(0, 0.2, 0.6, 1, 0.8, 1.5, 2.5, x)

pdf_gen_trap(0, 0.2, 0.6, 1, 0.7, 1.4, 0.8, x)

pdf_gen_trap(0, 0.2, 0.6, 1, 1, 3, 0.3, x)

pdf_gen_trap(0, 0.2, 0.6, 1, 0.7, 0.5, 2.2, x)
```
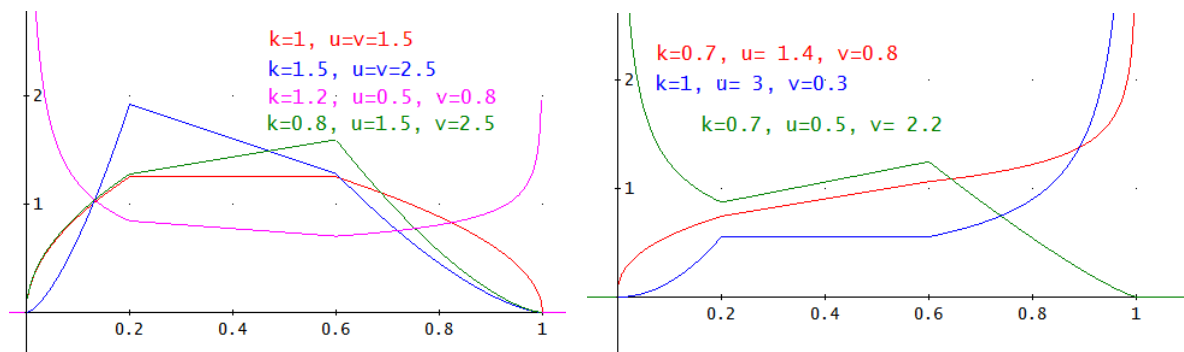

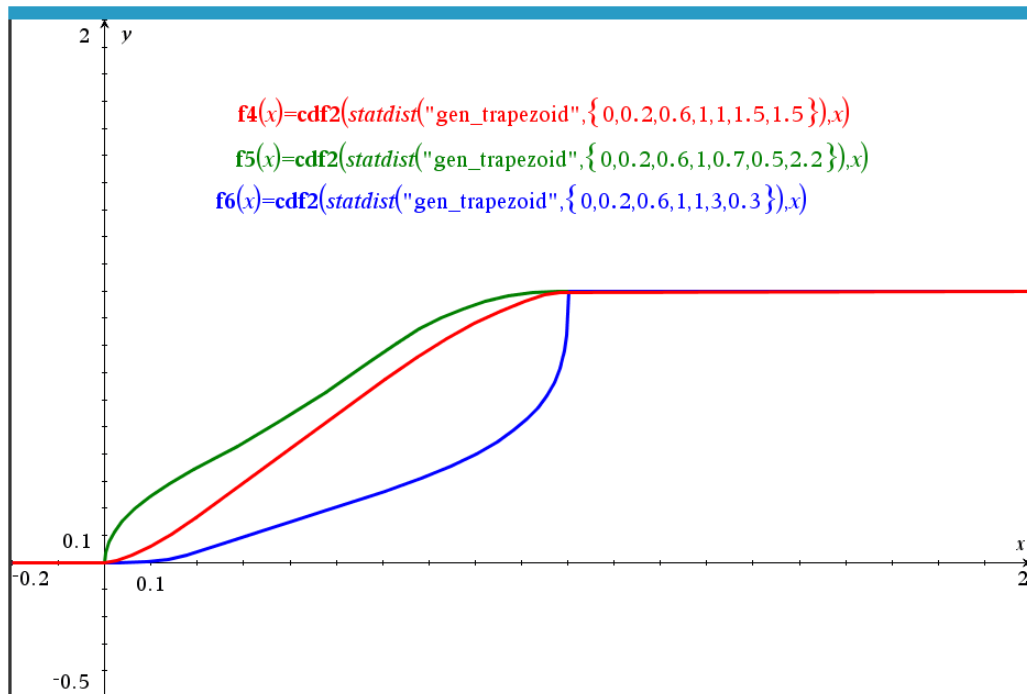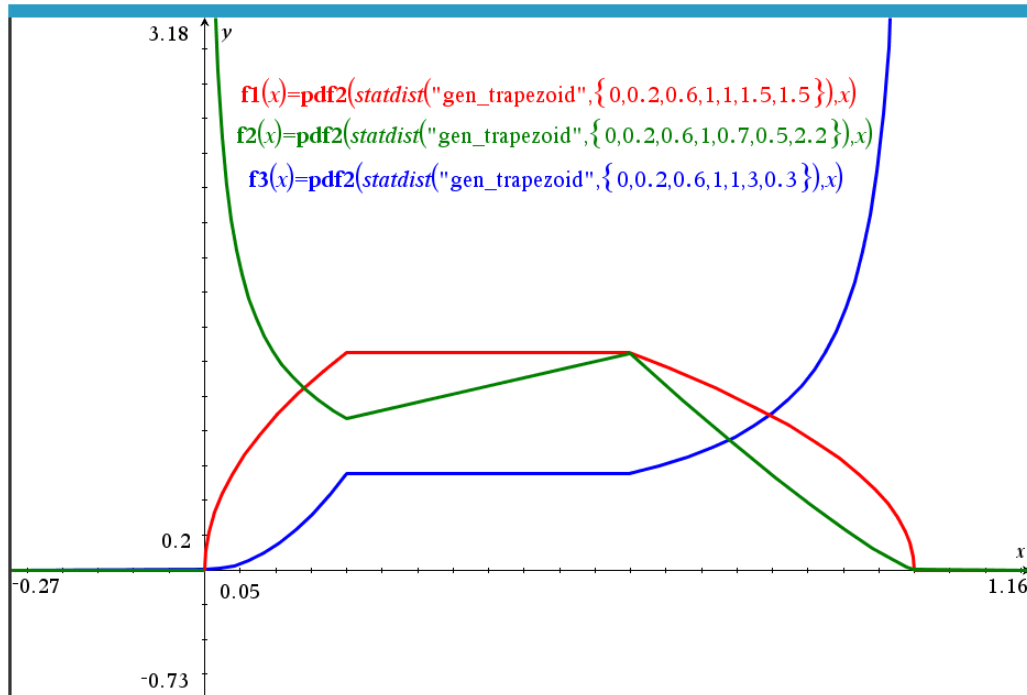Three integrals give the parts of the distribution function. And this is easy work thanks CAS!!

$$c1(a, m, n, b, k, u, v, x) := \int_a^x p1(a, m, n, b, k, u, v) \cdot f1(a, m, u, t)\, dt$$

$$c2(a, m, n, b, k, u, v, x) := c1(a, m, n, b, k, u, v, m) +$$
$$+ \int_m^x p2(a, m, n, b, k, u, v) \cdot f2(m, n, k, t)\, dt$$

$$c3(a, m, n, b, k, u, v, x) := c2(a, m, n, b, k, u, v, n) +$$
$$+ \int_n^x p3(a, m, n, b, k, u, v) \cdot f3(n, b, v, t)\, dt$$

```
cdf_gen_trap(a, m, n, b, k, u, v, x) :=
  If x < a
    0
    If x < m
      c1(a, m, n, b, k, u, v, x)
      If x < n
        c2(a, m, n, b, k, u, v, x)
        If x < b
          c3(a, m, n, b, k, u, v, x)
          1
```

Call and plot of three cdfs.

```
cdf_gen_trap(0, 0.2, 0.6, 1, 0.7, 1.4, 0.8, x)

cdf_gen_trap(0, 0.2, 0.6, 1, 1, 3, 0.3, x)

cdf_gen_trap(0, 0.2, 0.6, 1, 0.7, 0.5, 2.2, x)
```
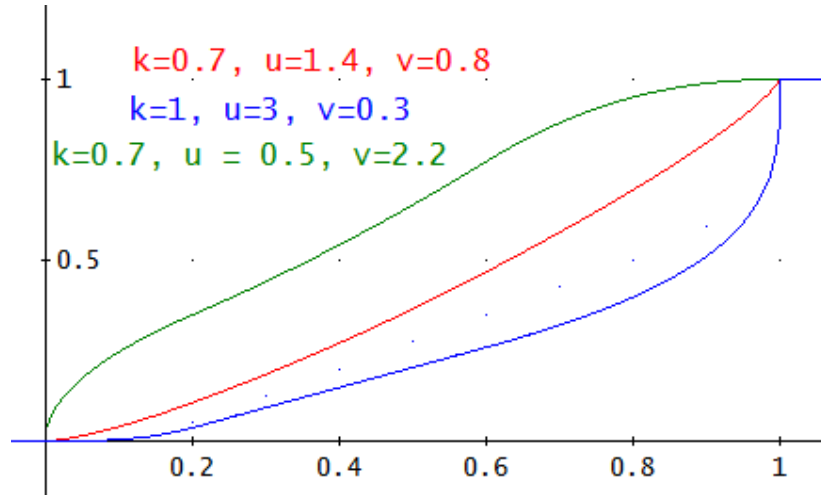

I could transfer the functions step by step to TI-NspireCAS and included pdf and cdf into functions pdf2 and cdf2 contained in the library mathtool.tns.

Please compare the plots of the pdfs and cdfs with the respective *Derive* plots on pages 42 and 43.

More links:

[1] https://www.researchgate.net/publication/225392327_Generalized_trapezoidal_distributions

https://cran.r-project.org/web/packages/trapezoid/vignettes/trapezoid.pdf

https://www.vosesoftware.com/riskwiki/GeneralizedTrapezoidUniform(GTU)distribution.php

https://www.sciencedirect.com/science/article/pii/S2590054420300245?dgcid=rss_sd_all
(Generalized trapezoidal distribution and Covid19 deaths)