Large permutations and permutons

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Introduction

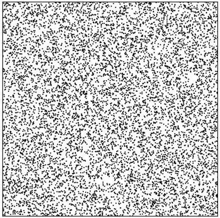
Main topic: random permutations

- Classical questions: look at some statistics, like the number of cycles (of given length), pattern occurrences, longest increasing subsequences, ...
  (usually for uniform, Ewens or Mallows distributions)

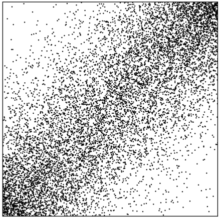
- a more recent approach: look for a limit for the rescaled permutation matrix; such limits are called permutons.
  (interesting for non-uniform models or constrained permutations)

This talk: *very biased* presentation of the notion of permutons and some literature on them.

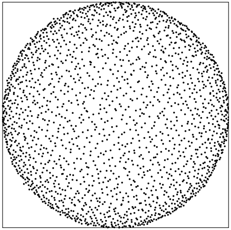
A few random permutations



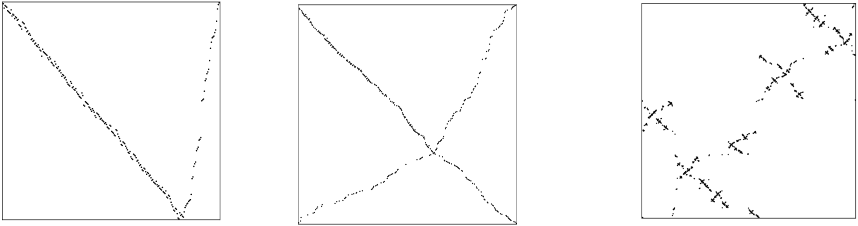
Uniform



Mallows ($\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$)



Sorting network,
half way (©AHRV '07)



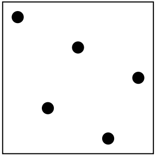
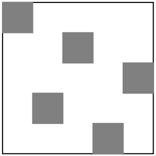
Uniform random pattern-avoiding permutations

The theory of permutons

(Hoppen, Kohayakawa, Moreira, Rath, Sampaio, '13)

How to look at large permutations?

A permutation π can be encoded as a probability measure μ_π on $[0,1]^2$.

$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

We have a natural notion of limit for such objects: the weak convergence.
This defines a nice compact Polish space.

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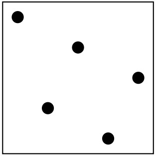
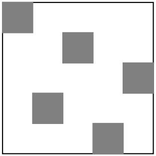
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

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Note: the projection on μ_π on each axis is the Lebesgue measure on $[0,1]$ (in other words, μ_π has uniform marginals).
$\rightarrow$ potential limits also have uniform marginals.

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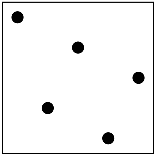
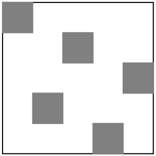
$\pi = 5\,2\,4\,1\,3 =$  $\mapsto \quad \mu_\pi =$ 

In μ_π, each small square has weight $1/n$ (i.e. density n).

Definition

A permuton is a probability measure on $[0,1]^2$ with uniform marginals.

Next few slides: connection with permutation patterns.

Permutation patterns

Definition

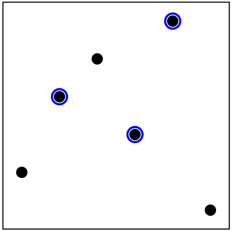
An occurrence of a pattern τ in σ is a subsequence $\sigma_{i_1} \dots \sigma_{i_k}$ that is order-isomorphic to τ, *i.e.* $\sigma_{i_s} < \sigma_{i_t} \Leftrightarrow \tau_s < \tau_t$.

Example (occurrences of 2 1 3)

2 4 5 3 6 1

8 2 3 4 6 1 7 5

Visual interpretation



Pattern density in permutations and permutons

If τ and σ are permutations of size k and n, resp., we set

$$\widetilde{\text{occ}}(\tau,\sigma) := \binom{n}{k}^{-1} \cdot \# \left\{ \begin{array}{c} \text{occurrences of} \\ \tau \text{ in } \sigma \end{array} \right\} \in [0,1].$$

In other terms: take k elements uniformly at random in σ, the probability to find a pattern τ is $\widetilde{\text{occ}}(\tau,\sigma)$.

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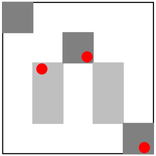
This probabilistic interpretation extends to permutons: replacing σ with a permuton μ

$$\widetilde{\text{occ}}(\tau,\mu) := \mathbb{P}^{\mu}(U^{(1)},\cdots,U^{(k)} \text{ form a pattern } \tau),$$

where $U^{(1)},\cdots,U^{(k)}$ are i.i.d. points in $[0,1]^2$ with distribution μ.



a "231 pattern" in a permuton

An approximation lemma

Reminder:

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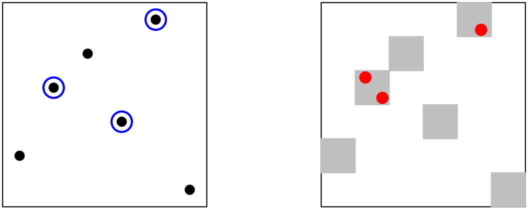
⚠ In general, $\widetilde{\mathrm{occ}}(\tau,\sigma) \neq \widetilde{\mathrm{occ}}(\tau,\mu_\sigma)$.



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But we have the following approximation lemma:

Lemma

If π and σ are permutations of size k and n, resp., then

$$|\,\widetilde{\text{occ}}(\pi,\sigma) - \widetilde{\text{occ}}(\pi,\mu_\sigma)| \leq \frac{1}{n}\binom{k}{2}.$$

Pattern density convergence and permuton convergence

Theorem (Hoppen, Kohayakawa, Moreira, Rath, Sampaio, 2013)

Weak convergence of permutons is equivalent to the pointwise convergence of $\widetilde{\mathrm{occ}}(\tau, \cdot)$ *for all* τ, i.e.

$$\mu^{(n)} \to \mu \;\Leftrightarrow\; \text{for all } \tau,\ \widetilde{\mathrm{occ}}(\tau, \mu^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu).$$

As a consequence, for a sequence of permutation $\sigma^{(n)}$ of size tending to infinity,

$$\mu_{\sigma^{(n)}} \to \mu \;\Leftrightarrow\; \text{for all } \tau,\ \widetilde{\mathrm{occ}}(\tau, \sigma^{(n)}) \to \widetilde{\mathrm{occ}}(\tau, \mu).$$

(In terms of permutations, $\widetilde{\mathrm{occ}}(\tau, \sigma^{(n)})$ is much more concrete!)

Permuton convergence of random permutations

Theorem (Bassino-Bouvel-F.-Gerin-Maazoun-Pierrot, 17)

Let $\boldsymbol{\sigma}_n$ be a random permutation of size n. The following assertions are equivalent.

(a) $\mu_{\boldsymbol{\sigma}_n}$ *converges in distribution for the weak topology to some random permuton $\boldsymbol{\mu}$.*

(b) *The random infinite vector $\big(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\big)_{\pi \in \mathfrak{S}}$ converges in distribution in the product topology to some random infinite vector $(\boldsymbol{\Lambda}_\pi)_{\pi \in \mathfrak{S}}$.*

(c) *For every π in $\mathfrak{S}$, there is a $\Delta_\pi \geq 0$ such that*

$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \xrightarrow{n \to \infty} \Delta_\pi.$$

Note: $(a) \Leftrightarrow (b)$ expected (random version of the previous result),
$(b) \Leftrightarrow (c)$ might be more surprising (cv in expectation is enough!).

A partial literature review on permutons

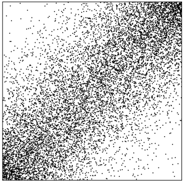
Limit permuton for Mallows permutations (Starr, '09)

Mallows model on S_n: $\mathbb{P}(\sigma_n) \propto q_n^{\mathrm{inv}(\sigma_n)}$,
where $\mathrm{inv}(\sigma) = \#\{(i,j) \text{ with } i < j \text{ and } \sigma(i) > \sigma(j)\}$.

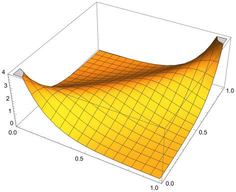
Theorem (Starr, '09)

Take $q_n = 1 - \beta/n$. Then $\mu_{\boldsymbol{\sigma}^{(n)}}$ converge to the deterministic permuton with density

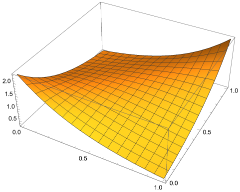
$$u(x,y) = \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\beta[x-y]/2) - e^{-\beta/4}\cosh(\beta[x+y-1]/2)\right)^2}.$$



Simulation ($n = 10000$, $\beta = 6$)



$\beta = 6$



$\beta = 2$

A large deviation principle

Definition (entropy of a permuton μ with density g)

$$H(\mu) = \int_{[0,1]^2} -g(x,y) \log g(x,y) dxdy \leq 0.$$

If μ has no density, $H(\mu) := -\infty$.

Theorem (Trashorras, '08, Kenyon, Král, Radin, Winkler, '15)

Let Λ be a set of permutons, Λ_n the set of permutations $\pi \in S_n$ with $\mu_\pi \in \Lambda$. Then:

1. *If Λ is closed,* $\limsup_{n\to\infty} \frac{1}{n} \log \frac{|\Lambda_n|}{n!} \leq \sup_{\mu \in \Lambda} H(\mu)$;
2. *If Λ is open,* $\liminf_{n\to\infty} \frac{1}{n} \log \frac{|\Lambda_n|}{n!} \geq \sup_{\mu \in \Lambda} H(\mu)$.

Informally, the number of permutations of size n close to a permuton μ is

$$n!\, e^{(H(\mu)+o(1))n}.$$

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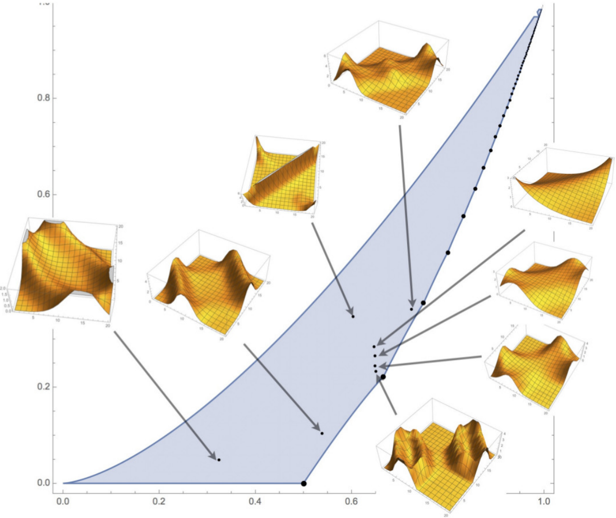
Q: which permutons maximize the entropy under some constraints? (such as fixing some pattern densities)

A nice picture (Kenyon, Král, Radin, Winkler, '15)



x-axis: $\widetilde{occ}(12, \mu)$
y-axis: $\widetilde{occ}(123, \mu)$

blue zone: zone where there exists a permuton μ with such pattern densities.

Displayed permutons are entropy maximizers for fixed 12 and 123 densities.

©KKRW, '15

And more. . .

- limit shape of Erdős-Szekeres permutations (i.e. permutations with a square RSK shape): limiting permuton supported by the interior of an explicit degree 4 algebraic curve (Romik '06).
- Random sorting networks (Angel, Holroyd, Romik, Virág, '06; Dauvergne '18) define some dynamics on permutations and permutons (Rahman, Virág, Vizer, '16).
- Mukherjee ('16): permuton limits of other biased random permutation models, convergence of number of cycles of fixed length in Mallows permutations.

Limits of permutation classes with a finite specification

(joint work with Bouvel, Bassino, Gerin, Maazoun, Pierrot, next week on arXiv)

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Permutation classes

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A set $\mathcal{C}$ of permutations (of all sizes) is a class if for all permutations π in $\mathcal{C}$, and all *patterns* τ of π, τ is also in $\mathcal{C}$.

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- More recently from a probabilistic point of view: what does a uniform random permutation in a given class look like?
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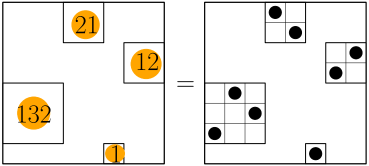
Note: large deviation theory does not apply.

Substitution in permutations (1/2)

Definition

Let θ be a permutation of size d and $\pi^{(1)}, \dots, \pi^{(d)}$ be permutations. The diagram of the permutation $\theta[\pi^{(1)}, \dots, \pi^{(d)}]$ is obtained by replacing the i-th dot in the diagram of θ with the diagram of $\pi^{(i)}$ (for each i).

$2413[132, 21, 1, 12] =$  $= 24387156$

Definition

A permutation is called *simple* if it cannot be obtained as a nontrivial substitution.

Examples: 12, 21, 3142, 2413, , 25314, , ...

Substitution in permutations (2/2)

Proposition (Albert, Atkinson, '05)

Every permutation σ of size $n \geq 2$ can be uniquely decomposed as either:

- $\alpha[\pi^{(1)}, \dots, \pi^{(d)}]$, *where α is simple of size $d \geq 4$,*
- $12[\pi^{(1)}, \pi^{(2)}]$, *where $\pi^{(1)}$ is 12-indecomposable,*
- $21[\pi^{(1)}, \pi^{(2)}]$, *where $\pi^{(1)}$ is 21-indecomposable.*

Not very interesting for uniform random permutation: the simple permutation α has typically size $n - O(1)$.

But interesting for permutations in classes! It has been used for enumerating many classes.

Classes with finitely many simple permutations (1/2)

Assume we have a finite number of simple permutations in a class $\mathcal{C}$.

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First thought: great, the substitution decomposition gives us a system of equation for the class

$$\begin{cases} \mathcal{C} \overset{?}{=} \{\bullet\} \uplus 12[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}] \uplus 21[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \uplus \left(\biguplus_{|\alpha| \geq 4} \alpha[\mathcal{C}, \dots, \mathcal{C}] \right) \\ \mathcal{C}^{\mathrm{not}\oplus} \overset{?}{=} \{\bullet\} \uplus 21[\mathcal{C}^{\mathrm{not}\ominus}, \mathcal{C}] \uplus \left(\biguplus_{|\alpha| \geq 4} \alpha[\mathcal{C}, \dots, \mathcal{C}] \right) \\ \mathcal{C}^{\mathrm{not}\ominus} \overset{?}{=} \{\bullet\} \uplus 12[\mathcal{C}^{\mathrm{not}\oplus}, \mathcal{C}] \uplus \left(\biguplus_{|\alpha| \geq 4} \alpha[\mathcal{C}, \dots, \mathcal{C}] \right). \end{cases}$$

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not quite, we can create forbidden patterns in the substitution!

$\rightarrow$ we need to replace some of the $\mathcal{C}$ above by some subfamilies of $\mathcal{C}$, consider cases, resolve ambiguities and iterate...

Classes with finitely many simple permutations (2/2)

Theorem (Bassino-Bouvel-Pierrot-Pivoteau-Rossin '17)

Any class $\mathcal{C}$ with finitely many simple permutations admits a finite combinatorial specification of the form

$$\mathcal{C}_i = \varepsilon_i\{\bullet\} \uplus \biguplus_{\alpha \in \mathcal{S}_{\mathcal{C}_i}} \biguplus_{(k_1,\dots,k_{|\alpha|}) \in K^i_\alpha} \alpha[\mathcal{C}_{k_1}, \cdots, \mathcal{C}_{k_{|\alpha|}}] \qquad (0 \le i \le d) \quad (1)$$

where the $\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1, \cdots, \mathcal{C}_d$ and the ε_i are in $\{0,1\}$.

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where the $\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1, \cdots, \mathcal{C}_d$ *and the* ε_i *are in* $\{0, 1\}$.

The system can be obtained algorithmically (implemented by Maazoun).

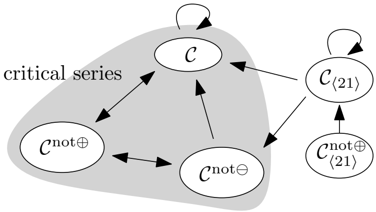
$\rightarrow$ gives an algebraic system of equations for the GF of $\mathcal{C}$.

$\rightarrow$ yields a random sampler for the class $\mathcal{C}$ (used for simulations in the introduction).

Finite specification: the example of Av(132)

$$\begin{cases} \mathcal{C} &= \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\text{not}\oplus}, \mathcal{C}_{\langle 21\rangle}] \quad \uplus \quad \ominus[\mathcal{C}^{\text{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\text{not}\oplus} &= \{\bullet\} \quad \uplus \quad \ominus[\mathcal{C}^{\text{not}\ominus}, \mathcal{C}] \\ \mathcal{C}^{\text{not}\ominus} &= \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\text{not}\oplus}, \mathcal{C}_{\langle 21\rangle}] \\ \mathcal{C}_{\langle 21\rangle} &= \{\bullet\} \quad \uplus \quad \oplus[\mathcal{C}^{\text{not}\oplus}_{\langle 21\rangle}, \mathcal{C}_{\langle 21\rangle}] \\ \mathcal{C}^{\text{not}\oplus}_{\langle 21\rangle} &= \{\bullet\}. \end{cases}$$

Associated dependency graph indicating families with maximal growth rate (called critical families):



Main theorem

Theorem (BBFGMP, '19)

Let $\mathcal{C}$ be a family of permutations with a finite analytic specification (e.g. a permutation class with finitely many simple permutations). Assume that the dependency graph restricted to critical families is strongly connected (plus some weak aperiodicity assumption).

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essentially linear case *If the specification contains no products of critical families, then a uniform random permutation in the class converges to an X-permuton with computable parameters.*

essentially branching case *If the specification contains a product of critical families, then a uniform random permutation in the class converges to a Brownian separable permuton with computable parameters.*

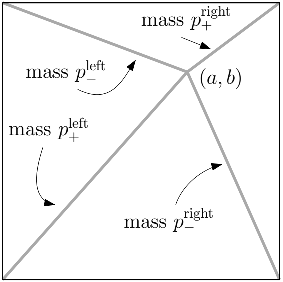
Description of the limit permutons and examples in the next few slides...

The X-permuton

Parameter: a quadruple of sum 1

$$(p_+^{\text{left}}, p_+^{\text{right}}, p_-^{\text{left}}, p_-^{\text{right}}).$$

We set $a = p_+^{\text{left}} + p_-^{\text{left}}$
and $b = p_+^{\text{left}} + p_-^{\text{right}}$
(to ensure the uniform marginal condition).



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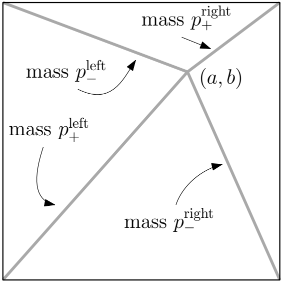
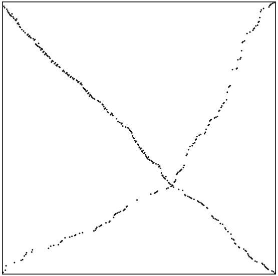
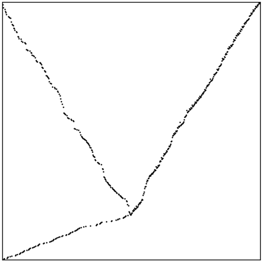


Note: this is a deterministic permuton. When random permutations converge to the X-permuton, we have a concentration phenomenon, i.e. two independent random permutations are closed to each other.

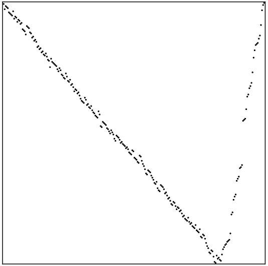
The essentially linear case: examples



$\mathrm{Av}(2413, 3142, 2143, 34512)$



$\mathrm{Av}(231, 21543)$

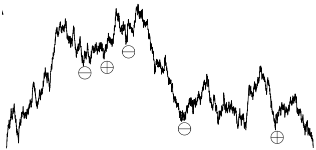


$\mathrm{Av}(2413, 1243, 2341, 41352, 531642)$

Note: in the second (resp. third) case, one (resp. two consecutive) parameters are 0. Diagonals are also degenerate X-permutons (with 2 opposite or 3 parameters equal to 0).

The Brownian separable permuton (Maazoun '17)

Parameter: $p \in [0,1]$



(e, S)

- e is a Brownian excursion and $S : \mathrm{LocalMin}(e) \to \{\oplus, \ominus\}$ is a independent assignment of signs to local minima of e (the probability to get a $\oplus$ is p).

The Brownian separable permuton (Maazoun '17)

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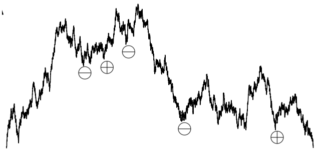
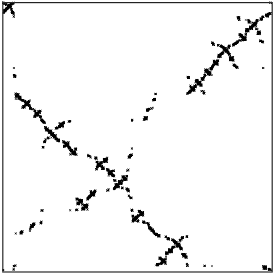


(e, S)

$\mapsto \;\; \sigma \;\; \mapsto$



$\boldsymbol{\mu} = (x, \sigma(x))_\star(\mathrm{Leb}([0,1])$

- $\sigma : [0,1] \to [0,1]$ is the unique Lebesgue preserving function s.t. (x,y) is an inversion if and only if the sign of $\min_{[x,y]} e$ is $\ominus$.
- The Brownian separable permuton is the "graph of the function σ".

The Brownian separable permuton (Maazoun '17)

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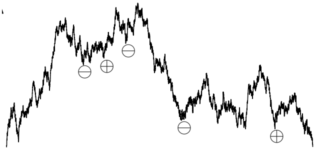

(e, S)

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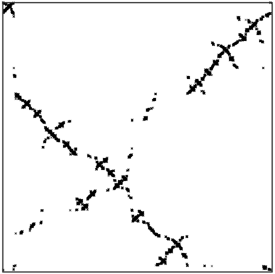
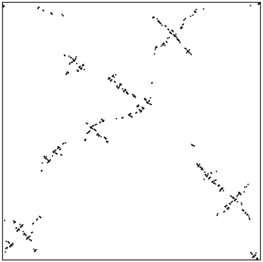


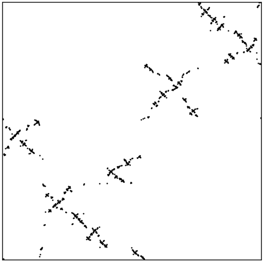
$\boldsymbol{\mu} = (x, \sigma(x))_{\star}(\mathrm{Leb}([0,1])$

Note: this a random permuton. No concentration phenomenon here.

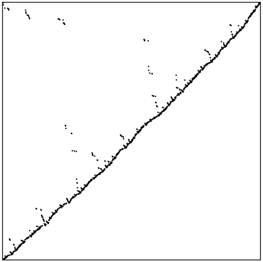
The essentially branching case: examples



Av(2413, 3142)
separable permutations



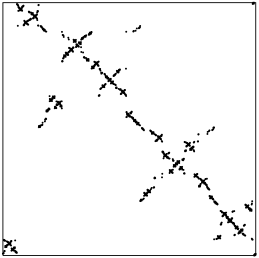
Av(2413, 31452,
41253, 41352, 531246)



Av(231)

The limit in the last case is a degenerate Brownian permuton with $p = 1$, that is the diagonal of the square. This convergence to the diagonal (and much more precise results) was already known.

The essentially linear case: an almost-example



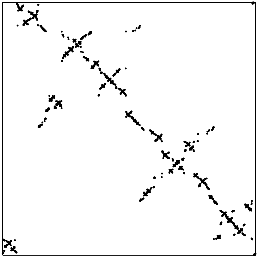
Doubly alternating Baxter permutations (©Dokos, Pak)

The main result of Dokos-Pak article is the limit

$$\mathbb{P}\big[\boldsymbol{\sigma}_n(\lfloor \alpha n \rfloor) = \lfloor \beta n \rfloor\big].$$

The question of studying of $\boldsymbol{\sigma}_n$ itself (and this picture) is in the open problem section.

The essentially linear case: an almost-example



Doubly alternating Baxter permutations (©Dokos, Pak)

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$$\mathbb{P}\big[\boldsymbol{\sigma}_n(\lfloor \alpha n \rfloor) = \lfloor \beta n \rfloor\big].$$

The question of studying of $\boldsymbol{\sigma}_n$ itself (and this picture) is in the open problem section.

Our result does not apply as is because of periodicity issues, but proving the convergence to the Brownian permuton should not be difficult.

A word on the proofs

1. Reminder: enough to prove that, for any τ,

$$\mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau, \boldsymbol{\sigma}_n)\big] \to \mathbb{E}\big[\,\widetilde{\mathrm{occ}}(\tau, \boldsymbol{\nu})\big],$$

   where $\boldsymbol{\nu}$ is the targeted limit random permuton.

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   where $\boldsymbol{\nu}$ is the targeted limit random permuton.

2. The RHS can be evaluated easily (elementary for X-permuton, using some results on Brownian excursion for the Brownian one).

3. The LHS can be computed combinatorially:

$$\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] = \frac{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \mathrm{pat}_I(\sigma) = \pi\}}{\binom{n}{k}\ |\mathcal{C}_n|}.$$

   We will estimate that through analytic combinatorics.

Analytic combinatorics

The strongly connectedness hypothesis ensures that

- in the essentially linear case,

$$C(z) \sim a \frac{1}{1-\frac{z}{\rho}}, \text{ implying } |\mathcal{C}_n| \sim a\rho^{-n}.$$

- in the branching case,

$$C(z) \sim a - b\sqrt{1-\tfrac{z}{\rho}}, \text{ implying } |\mathcal{C}_n| \sim \frac{b}{2\sqrt{\pi}} n^{3/2} \rho^{-n}$$

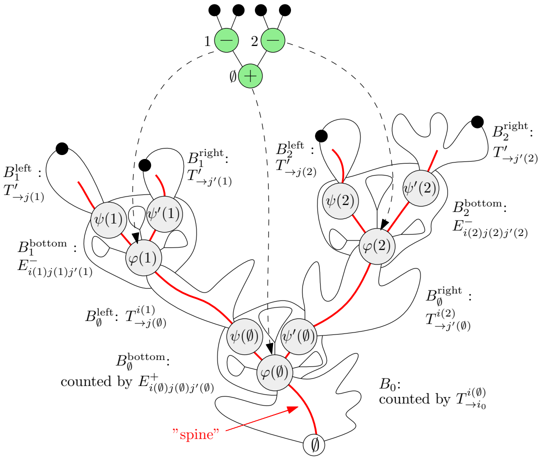
The difficulty is to estimate

$$\{\#\{\sigma \in \mathcal{C}_n, I \subset [n] : \mathrm{pat}_I(\sigma) = \pi\}\}.$$

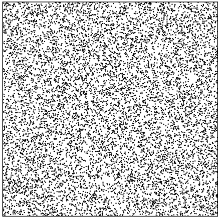
We need to write some equations for the corresponding generating function and to find the behavior at the singularity.

A picture of a combinatorial decomposition

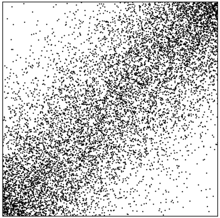
(where permutations are encoded by trees thanks to the specification.)



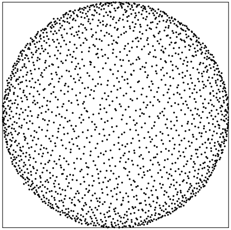
Thank you for your attention



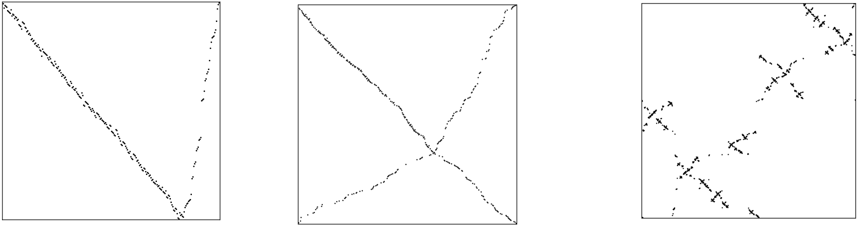
Uniform



Mallows ($\mathbb{P}(\sigma) \propto q^{\mathrm{inv}(\sigma)}$)



Sorting network,
half way (©AHRV '07)

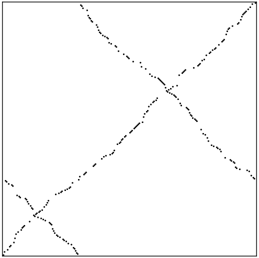


Uniform random pattern-avoiding permutations

Extra slide 1: is the strong connectivity condition necessary?

Yes!

Here is a class with finitely many simple permutations and a "double X" limit:



Av(214365, 3412, 52143, 32541)

We can treat such examples on a case-by-case basis from their finite specification, but we have no general theorem!

Extra slide 2: the *intensity* of the Brownian permuton

Since the Brownian permuton $\boldsymbol{\mu}_p$ is a random measure, we can consider its intensity measure $\mathbb{E}\boldsymbol{\mu}_p$, defined by

$$(\mathbb{E}\boldsymbol{\mu}_p)(R) = \mathbb{E}(\boldsymbol{\mu}(R)), \text{ for any rectangle } R \subseteq [0,1]^2.$$

Theorem (Maazoun '17)

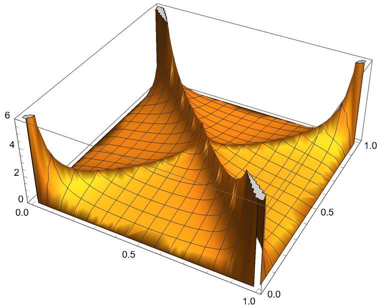
The intensity measure $\mathbb{E}\boldsymbol{\mu}_p$ has density w.r.t to Lebesgue measure

$$f_p(x,y) = \int_{\max(0,x+y-1)}^{\min(x,y)} \frac{3p^2(1-p)^2 da}{2\pi(a(x-a)(1-x-y+a)(y-a))^{3/2}\left(\frac{p^2}{a}+\frac{(1-p)^2}{(x-a)}+\frac{p^2}{(1-x-y+a)}+\frac{(1-p)^2}{(y-a)}\right)^{5/2}}.$$

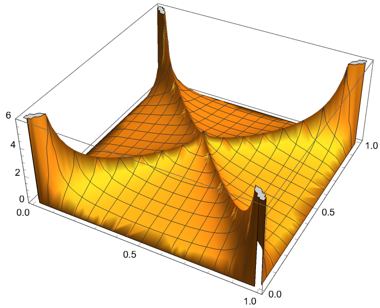
Concretely, if $\boldsymbol{\sigma}_n$ tends to $\boldsymbol{\mu}_p$, then, for any rectangle $R \subseteq [0,1]^2$

$$\mathbb{E}\big[\#\{(i,j) \in nR : \sigma(i) = j\}\big] \sim n \int_{(x,y)\in R} f_p(x,y) dxdy.$$

Extra slide 2bis: picture of $\mathbb{E}\boldsymbol{\mu}_p$



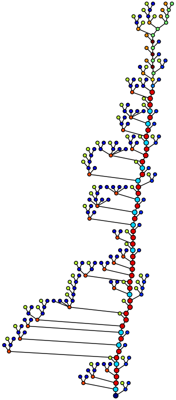
density of $\mathbb{E}\boldsymbol{\mu}_{.4}$



density of $\mathbb{E}\boldsymbol{\mu}_{.5}$

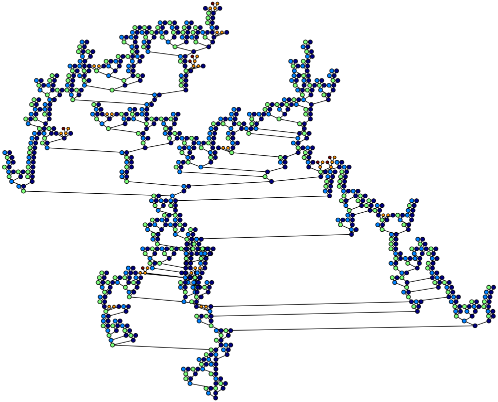
For $p = .5$, this function was found (under a different form) by Pak and Dokos, in the context of doubly alternating Baxter permutations.

Extra slide 3: underlying random trees



essentially linear case

Av(2413, 1243, 2341, 41352, 531642)



essential branching case

Av(2413, 31452, 41253, 41352, 531246)