Toward Finitist Proof Theory

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March 24, 1998

Technical Report No. CMU-PHIL-90

Philosophy

Methodology

Logic

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0. Introduction. A little more than three quarters of a century ago, in the Spring of 1921, Hilbert gave the first presentation of his new investigations concerning the foundations of arithmetic in Kopenhagen and some months later in Hamburg. Hilbert's 1922 paper *Neubegründung der Mathematik* is based on these talks.[2] The paper is important for a variety of systematic reasons, but also for its opening combative remarks against Weyl and Brouwer. It is remarkable, how little is known about the intellectual development that led to the logical work presented there; it is similarly remarkable, how little is known about the intellectual climate that provoked Hilbert's remarks. There is a personal edge to these remarks; after all, they are in part directed to Weyl who had been Hilbert's own student and who had joined the "revolutionary" movement of Brouwer's intuitionism in 1920. Weyl had also published in 1921 the unabashedly "propagandistic" paper *Über die neue Grundlagenkrise der Mathematik*; note that "revolutionary" and "propagandistic" are Weyl's characterizations, not mine. This was the unfortunate beginning of the unfortunate "Grundlagenstreit" in the twenties that, even more unfortunately, still colors our views on issues in the foundations of mathematics. Here is a rich mine for fascinating historical, logico-mathematical, and philosophical investigations that can make substantive contributions to the contemporary discussion in the philosophy of mathematics. In my remarks I am going to focus on the intellectual developments within the Hilbert School.[3]

During the last ten or fifteen years a multi-faceted perspective on the work of the Hilbert School has been emerging. That has been achieved mainly by bringing out the rich context in which the work is embedded: important connections have been established, on the one hand, to foundational work of the 19th century (that had been viewed as largely irrelevant) and, on the other hand, to a general reductive program (that evolved out of Hilbert's and underlies implicitly most modern proof theoretic investigations). However, it is crucial to gain a better understanding

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[2] Thus, what more appropriate place than Kopenhagen to reflect on proof theory and Hilbert's broader foundational investigations, but also on the work of those whom Hilbert criticized so sharply? -- I am grateful to the organizers, in particular Professor Pedersen and Dr. Hendricks, for bringing this workshop to life and providing the opportunity for such reflection!

[3] There are many historical questions I don't know how to answer; some are mentioned in footnotes and for particularly important ones I indicated what partial information I have. For additional information concerning the period before 1917 see Abrusci and Peckhaus; Moore gives some information about the development starting in 1917/18. As to a broader discussion of Hilbert's investigations, see the papers by Feferman, Hallett, Sieg, and Stein. -- Many of the relevant original texts have finally been translated into English and can be found in (Ewald) and (Mancosu).

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of the development of Hilbert's thought on the foundations of arithmetic, where arithmetic is understood in a broad sense that includes elementary number theory and reaches all the way to set theory. Admittedly, this is just one aspect of Hilbert's work on the foundations of mathematics, as it disregards the complex interactions with his foundational work on geometry and the natural sciences; it is a most significant aspect though, as it reveals a surprising internal dialectic progression and throws a distinctive light on the origins of modern mathematical logic.

My paper is focused on developments between 1917 and 1922, a period of very special interest. Standard wisdom partitions Hilbert's work on the foundations of arithmetic, with some justification, into two periods (and suggests that the time in between was devoted to important other matters). The first period is taken to extend from 1900 to 1905, the second from 1922 to 1931. The periods are marked by dates of outstanding publications. Hilbert published in 1900 and 1905 respectively *Über den Zahlbegriff* and *Über die Grundlagen der Logik und Arithmetik*. The considerations of the latter paper, according to this view, were taken up around 1920, were quickly expanded into the proof theoretic program and exposed in 1922 through Hilbert's *Neubegründung der Mathematik* and Bernays' *Über Hilberts Gedanken zur Grundlegung der Arithmetik*. Finally, it is argued that the pursuit of the program was halted in 1931 by Gödel's paper *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*.

This partition of Hilbert's work does not include, or accomodate easily, the programmatic paper *Axiomatisches Denken* published in 1918. The paper had been presented already in September 1917 to the Swiss Mathematical Society in Zürich and advocates a *logicist reduction of mathematics*.[4] In sharp contrast, the 1922 papers by Hilbert and Bernays set out the philosophical and mathematical-logical goals of *the* Hilbert Program. This remarkable progression is not at all elucidated by publications, but it can be

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[4] There is clear evidence of Hilbert's growing familiarity with Russell's work, starting in 1913/4; what I know is presented in an Appendix to my (1997A). Bernays, in his (1935) minimizes the programmatic logicist direction of Hilbert's thinking. He also encourages the "standard view" of the development toward Hilbert's Program by describing it as follows: "In diesem vorläufigen Stadium hat Hilbert [1905; WS] seine Untersuchungen über die Grundlagen der Arithmetik für lange Zeit unterbrochen. Ihre Wiederaufnahme finden wir angekündigt in dem 1917 gehaltenen Vortrag 'Axiomatisches Denken'. ... Dem hiermit von neuem gefaßten Plan einer Beweistheorie hat sich Hilbert in den nachfolgenden Jahren, insbesondere seit 1920, vornehmlich gewidmet. Ein verstärkter Antrieb hierzu erwuchs ihm aus der Opposition, welche Weyl und Brouwer gegen das übliche Verfahren der Analysis und Mengenlehre richteten." In a footnote to this remark Bernays lists the papers (1918), (1919), (1919A), and (1921) by Brouwer and (1918), (1919), and (1921) by Weyl.

analysed by reference to notes for lectures Hilbert gave during that period in Göttingen. It is this progression I want to depict. But before doing that in parts 2 through 4, I recall very briefly some pertinent context.

1. BACKGROUND. Hilbert viewed the *axiomatic method* as holding the key to a systematic organization of any sufficiently developed subject; he also saw it as providing the basis for metamathematical investigations of independence and completeness issues and for philosophical reflections. However, *consistency* was Hilbert's central concern ever since he turned his attention to the foundations of analysis in the late nineties of the last century. These 19-th century roots of Hilbert's work are very important and reveal the major intellectual forces that led Hilbert to the initial formulation of a *syntactic* consistency program in 1904/5.[5] As to the period from 1905 to 1917, I emphasize that Hilbert gave lecture courses on the foundations of mathematics almost every single year. We have notes for most of them; some are written out meticulously, for example by Max Born and Richard Courant. These lectures do not break new ground, in particular, they do not push along the "proof theoretic" approach of Hilbert's 1905 paper (that was so severely, yet fairly criticized by Poincaré in 1905/6). On the contrary, the notes for his *Set Theory* course in the summer term of 1917 and the almost contemporaneous paper *Axiomatisches Denken* reveal a logicist direction in Hilbert's work. In the notes Hilbert gives an axiom system for natural numbers and remarks that this is only a first step for his foundational investigation:

... if we set up the axioms of arithmetic, but forego their further reduction and take over uncritically the usual laws of logic, then we have to realize that we have not overcome the difficulties for a first philosophical-epistemological foundation; rather, we have just cut them off in this way.

In the essay *Axiomatisches Denken* he presses the issues further; viewing the examination of consistency as an "unavoidable task" he remarks:

... thus, it seems to be necessary to axiomatize logic itself and to *show that number theory as well as set theory are just parts of logic*. This avenue, prepared for a long time, not least by the deep investigations of Frege, has finally been taken most successfully by the penetrating mathematician and logician Russell. The completion of this broad Russellian enterprise of axiomatizing logic might be viewed quite simply as the crowning achievement of the work of axiomatization.

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[5] Let me mention stenographically: Dedekind, consistency concerns and semantic argument (1888) and (1890); Kronecker, emphasis on a thoroughly constructive approach; Cantor, letters to Hilbert communicating the inconsistency of Dedekind's framework (1897); Hilbert, from semantic argument to syntactic approach (1900; 1904/5). These connections are discussed in my papers (1990) and (1997A).

At the end of the set theory notes Hilbert emphasizes that, if we try to achieve such a reduction to logic, we are facing one of the most difficult problems of mathematics; he continues:

Poincaré has even the view that this is not at all possible. But with that view one could rest content only if it had been proved that the further reduction of the axioms for arithmetic is impossible; but that is not the case. Next term I hope to be able to examine more closely a foundation for logic.

One has the sense that the exigencies of academic life and the complexity of the issues diverted Hilbert's attention to his own great dissatisfaction. That motivated, I assume, Hilbert's action in the fall of 1917: he invited Paul Bernays to assist him in efforts to examine the foundations of mathematics. Bernays returned to Göttingen where he had been a student and started to work with Hilbert on lectures that were offered in the winter term 1917/18 under the title *Prinzipien der Mathematik*. The collaboration of Hilbert and Bernays led to a remarkable sequence of lectures. *Prinzipien der Mathematik* is the very first lecture in this sequence; the others are: *Logik-Kalkül* (winter term 1920)[6], *Probleme der mathematischen Logik* (summer term 1920; with Schönfinkel), *Grundlagen der Mathematik* (winter term 1921/22), and *Logische Grundlagen* (winter term 1922/23). The notes from lectures before the winter term of 1917/18 do not give any indication of what is to come; even the most attentive reader is quite unprepared for the full-blown creation of modern mathematical logic in the notes to be discussed now.

2. MATHEMATICAL LOGIC. The notes for *Prinzipien der Mathematik* consist of 246 type-written pages and are divided into two parts. Part A, *Axiomatische Methode*, gives on sixty-two pages Hilbert's standard account of the axiomatic method, in particular as it applies to geometry. Part B, *Mathematische Logik*, is a beautifully organized, almost definitive presentation of the very core of modern mathematical logic. (This part of the notes is, incidentally, a polished manuscript of Hilbert & Ackermann's 1928 book; the structure is taken over, and large sections of the book are identical with parts of the notes.) The detailed pursuit of the logicist goal required the presentation of a formal language for capturing the logical form of informal statements, the use of a formal calculus for representing the structure of logical arguments, and the

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[6] Because of the end of the First World War and soldiers having returned to the university, an extra semester was pressed into those two years: there was a "Zwischensemester" in 1919 (from September 22 to December 20); that was followed by the winter term 1920 and then by the regular summer term 1920, beginning on April 26.

formulation of "logical" principles for defining mathematical objects. This is carried through with remarkable focus, elegance, and directness. From the very beginning, the logical and mathematical questions are driven by philosophical reflections on the foundations of mathematics. The material is organized under five chapter headings: *The sentential calculus, The predicate calculus and class calculus* [i.e., monadic logic], *Transition to the function calculus* [i.e., full first order logic], *Systematic presentation of the function calculus* and, finally, *The extended function calculus*.

The first four chapters lead, in part, to a systematic formulation of first order logic; every step taken in expanding the logical framework is semantically motivated and carefully argued for. This material was novel at the time; by now it is all too familiar and will not be discussed except to note and emphasize one important difference: the languages contain *sentential* and *function* (i.e., relation) *variables*. The last chapter takes a noteworthy turn. If only a formalization of logical reasoning were aimed for, no additional work beyond that of the earlier chapters would be needed. However, the logical calculus is to play an important role for the investigation of mathematical theories and their relation to logic:

Not only do we want to develop individual theories from their principles in a purely formal way, but we also want to investigate the foundations of the mathematical theories and examine, what their relation to logic is and how far they can be built up from purely logical operations and concepts; and for this purpose the logical calculus is to serve as an auxiliary tool.

Detailed reflection leads "in the most natural way" to ramified type theory together with Russell's axiom of reducibility; this framework is then used for the development of analysis. The notes end with the remark:

Thus it is clear that the introduction of the axiom of reducibility is the appropriate means to turn the ramified calculus into a system out of which the foundations for higher mathematics can be developed.[7]

The notes present more than a system for the development of parts of higher mathematics. They constitute literally the first text presenting the core of modern logic with its distinctive metamathematical turn: the careful presentation of syntax and semantics provides the basis for the investigation of completeness and consistency issues, as they are now standardly examined in any first introduction to mathematical logic.

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[7] However, further discussions of the axiom of reducibility become increasingly critical and lead Hilbert and Bernays ultimately to the rejection of the logicist enterprise in 1920. These discussions sound explicitly and most clearly themes that are found in the literature, with equally good sense and balance, only in Gödel's paper on "Russell's Mathematical Logic".

The formal frame I have been discussing is not only contentually motivated, but its semantics is properly specified and the central semantic notions are carefully formulated. First order theories are always viewed together with suitable non-empty domains, *Bereiche*, indicating the range of the individual variables of the theory and the interpretations of the non-logical vocabulary (except, for sure, the sentential and function variables). In modern terms, they are always presented together with a *structure*. How are expressions of the formal language to be understood, given the associated domain? After the discussion of the axiom system for the function calculus there is the following remark clarifying where a semantic understanding is needed and where pure formality is essential:

This system of axioms provides us with a procedure to carry out logical proofs strictly formally, i.e., in such a way that we need not be concerned with the meaning of the judgements that are represented by formulas, rather we just have to attend to the prescriptions contained in the rules. However, we have to interpret the signs of our calculus when representing symbolically the premises from which we start and when understanding the results obtained by formal operations.

The logical signs are interpreted ... corresponding to the given linguistic reading; and the occurrence of indeterminate statement-signs and function-signs in a formula is to be understood as follows: for arbitrary replacements by determinate statements and functions the claim that results from the formula is correct.

The underlying concept of correctness, *Richtigkeit*, with respect to a domain is understood as follows: (1) statements involving no sentential or function variables are "correct" if they are true in the domain (and that is informally taken in exactly the same way as in the model theoretic arguments for independence and relative consistency in Hilbert's *Grundlagen der Geometrie* and in Gödel's 1929 dissertation); (2) if a statement does contain such variables, then the clause "for arbitrary replacements ..." is invoked to define "correctness" for this broader class of statements. Having clarified the basic semantics, I turn to completeness.

For the very purpose of the calculus in the systematic investigation it is crucial that the ordinary forms of logical argumentation can be recaptured formally. This is clearly expressed in the 1917/18 lecture notes[8]:

As for any other axiomatic system, one can raise also for this system the questions concerning consistency, logical dependencies, and completeness. The most important question is here that concerning completeness. After all, the goal of symbolic logic is to develop ordinary logic from the formalized assumptions. Thus it is essential to show that our axiom system suffices for the development of ordinary logic.

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[8] Such metamathematical concerns for logic are found already in the very early notes, e.g., of 1905.

The notes contain prominently only one precise concept of completeness for logical calculi, namely what has come to be known as *Post-completeness*: a calculus is Post-complete just in case "the addition of a formula, hitherto unprovable, to the system of basic formulas always leads to an inconsistent system". That is quickly established for sentential logic; in a footnote[9] the *semantic completeness* is proved. The latter notion is brought to the fore in Bernays' Habilitationsschrift of 1918 where the completeness theorem receives its first "classical" formulation, namely, "Every provable formula is a valid (allgemeingültig) formula and vice versa." For first order logic the question of Post-completeness is also raised, and it is conjectured that the answer is negative. The proof of this fact is given in Hilbert & Ackermann.

Excursion. The semantic completeness for first order logic is formulated as an open problem only in Hilbert & Ackermann's book: "Whether the axiom system is complete at least in the sense that really all logical formulas, that are correct for all domains of individuals, can be derived from it is an unsolved question. We can only say purely empirically that this axiom system has always sufficed for any application." Some recent commentators have viewed this formulation as *oddly obscure* (Goldfarb) or even *circular* (Dreben & van Heijenoort). Those views rest on a very particular reading of "logical formulas" that is narrowly correct, as Hilbert and Ackermann follow the 1917/18 notes verbatim and define them as those formulas that (i) do not contain symbols for determinate individuals and functions, and (ii) can be proved by appealing only to the logical axioms. Under this reading the formulation is indeed close to non-sensical. However, if one takes into account that "logische Formel" and "logischer Ausdruck" are used repeatedly in the book also as indicating formulas satisfying just (i), then their formulation together with the explication of correctness I reviewed earlier is right: the statement of the completeness problem involves *precisely* the definition of "allgemeingültig" found in Gödel's dissertation.

Let me come back to the 1917/18 notes and turn to consistency. Hilbert and Bernays exploit the standard arithmetic interpretation of the logical connectives to address the consistency problem; they show for both sentential and first order logic (by induction on derivations) that every provable formula is identically zero. Consistency of the logical calculi is a direct

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[9] on p. 153.

consequence.[10] In a note the reader is warned not to overestimate the significance of this result, because "[i]t does not give us a guarantee that the system of provable formulas remains free of contradictions after the symbolic introduction of contentually correct assumptions".[11] That much more difficult problem has to be attacked in special ways, perhaps by a logicist reduction or by quite new ways of proceeding. Notice that up to now no specifically proof theoretic considerations for the consistency problem have been mentioned. Indeed, the development toward the Hilbert Program as we think of it was completed only in the lectures given in the winter term 1921/22. Hilbert arrived at its formulation after abandoning the logicist route through two quite distinct steps, and only the second step takes up the earlier suggestion of developing a theory of (formal) proofs.

3. CONSTRUCTIVE NUMBER THEORY. The first step is taken in the winter term 1920. Hilbert reviews the logical matters from the 1917/18 lectures in a polished form, frequently referring back to them for additional details. The last third of the notes is devoted, however, to a completely different topic and develops number theory from a radically constructive point of view. Hilbert argues that the set theoretic or logical developments of Dedekind and Frege did not succeed in establishing the consistency of ordinary number theory and concludes:

To solve this problem I don't see any other possibility, but to rebuild number theory from the beginning and to shape concepts and inferences in such a way that paradoxes are excluded at the outset and that proof procedures become completely surveyable.
Now I will show how I think of the beginning of such a foundation for number theory.

The considerations are put back into the broader context of the earlier investigations, emphasizing once more the semantic underpinnings for axiom systems:

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[10] This is done on pages 70 ff and 150 ff; the analogous considerations are contained in *Hilbert & Ackermann* on pages 30 ff and 65 ff.

[11] on p. 156: Man darf dieses Ergebnis in seiner Bedeutung nicht überschätzen. Wir haben ja damit noch keine Gewähr, dass bei der symbolischen Einführung von inhaltlich einwandfreien Voraussetzungen das System der beweisbaren Formeln widerspruchslos bleibt.
In *Hilbert and Ackermann* there is a significant expansion of this remark:
Man darf das Ergebnis dieses Beweises für die Widerspruchsfreiheit unserer Axiome übrigens in seiner Bedeutung nicht überschätzen. Der angegebene Beweis der Widerspruchsfreiheit kommt nämlich darauf hinaus, daß man annimmt, der zugrunde gelegte Individuenbereich bestehe nur aus einem einzigen Element, sei also endlich. Wir haben damit durchaus keine Gewähr, daß bei der symbolischen Einführung von inhaltlich einwandfreien Voraussetzungen das System der beweisbaren Formeln widerspruchsfrei bleibt. Z.B. bleibt die Frage unbeantwortet, ob nicht bei der Hinzufügung der mathematischen Axiome in unserem Kalkül jede beliebige Formel beweisbar wird. Dieses Problem, dessen Lösung eine zentrale Bedeutung für die Mathematik besitzt, läßt sich in bezug auf Schwierigkeit mit der von uns behandelten Frage garnicht vergleichen. Die mathematischen Axiome setzen gerade einen unendlichen Individuenbereich voraus, und mit dem Begriff des Unendlichen sind die Schwierigkeiten und Paradoxien verknüpft, die bei der Diskussion über die Grundlagen der Mathematik eine Rolle spielen. (pp. 65-6)

We have analysed the language (of the logical calculus proper) in its function as a universal instrument of human reasoning and laid open the mechanism of logical argumentation.

However, the kind of viewpoint we have taken is incomplete in so far as the application of the logical calculus to a particular domain of knowledge requires an axiom system as its basis. I.e., one system (or several systems) of objects must be given and between them particular relations with particular assumed basic properties are considered.

This method is perfectly appropriate, Hilbert continues, when we are trying to obtain new results or present a particular science systematically. However, mathematical logic pursues also the goal of securing the foundations of mathematics.

For *this* purpose it seems appropriate to connect the mathematical constructions to what can be concretely exhibited and to interpret the mathematical inference methods in such a way that one stays always within the domain of what is controllable. And obviously one is going to start with arithmetic, as one finds here the most simple mathematical concepts.

In addition, it has been the endeavor in mathematics for a long time to reduce all conceptual systems (geometry, analysis) to the integers.

This remark is followed by the development of what might be called *strict finitist number theory*. The considerations are delicate, but one thing is perfectly clear: here is a version of constructive arithmetic stricter than what will appear a little later as finitist mathematics; it is stricter, because the directly meaningful part consists only of *closed* numerical equations. Bernays pointed to an *evolution toward* finitist mathematics at a number of places; in (1954) he wrote for example: "Originally, Hilbert intended to take the narrower standpoint that does not assume the *intuitive general concept of numeral*. That can be seen, for example, from his Heidelberg lecture (1904). It was already a kind of compromise that he accepted the finitist standpoint as presented in his publications." The narrower standpoint had been taken by Hilbert in some contexts during the period between 1905 and 1917, and in those contexts he scolds Kronecker for not being radical enough. Indeed, such remarks run like a minor red thread through the earlier notes and connect up with the 1905 paper; it is worthwhile to recall that Hilbert, under the impact of the elementary contradictions in set theory discovered by Zermelo and Russell, "temporarily thought that Kronecker had been probably right there. [That is, right in insisting on restricted methods; WS.] But soon he changed his mind. Now it became his goal, one might say, to do battle with Kronecker with his own weapons of finiteness by means of a modified conception of mathematics". These remarks were made by Bernays and are recorded in Constance Reid's biography of Hilbert.

In the lectures from the winter term 1920 this intuitive general concept of numeral is not yet assumed; instead, general statements like x+y=y+x are given a constructive and extremely rule-based interpretation:

Such an equation ... is not viewed as a claim for all numbers, rather it is interpreted in such a way that its full meaning is given by a proof procedure: each step of the procedure is an action that can be completely exhibited and that follows fixed rules.

Hilbert points out that, as a consequence of this view, the classical logical relations between general and existential statements do not obtain. After all, the truth of a general statement is usually equivalent to the non-existence of a counterexample. Under the given constructive interpretation the alternative between a general statement and the existence of a counterexample would be evident only with the additional assumption "Every equation without a counterexample is provable from the assumed arithmetic principles", as the meaning of the general statement depends on the underlying system of inference rules. The lecture notes conclude with a judicious statement in which Brouwer's name appears for the very first time:

This consideration helps us to gain an understanding for the meaning of the paradoxical claim, made recently by Brouwer, that for infinite systems the law of the excluded middle (the "tertium non datur") loses its validity.[12]

It must have been a discouraging conclusion for Hilbert to see so very clearly that this approach could not secure the foundations of classical mathematics either. However, he overcame the setback by taking a second strategic step in the lectures for the summer term 1920 that joined the considerations concerning a thoroughly constructive foundation of number theory with detailed formal logical work.

Already in his Heidelberg talk of 1904 and again in his Zürich lecture of 1917 Hilbert had argued for a "Beweistheorie", but had not pursued his suggestion systematically. Here, in section 7 of the notes from the summer term 1920, we find a consistency proof for an extremely restricted, quantifier-free part of elementary number theory that involves negations only as applied to equations. These considerations are based on Hilbert's 1905 paper and form the first part of Hilbert's *Neubegründung der Mathematik*. The latter paper's second part expands the basic framework in new ways. Bernays pointed to this "break" repeatedly and describes the paper's first part, for

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[12] The obvious historical question here is, what did Hilbert know about Brouwer's views. Bernays mentions in his (1935) a number of papers; cf. notes 4 and 14. -- One should recall in this context that in 1919 Brouwer had been offered a professorship in Göttingen. More precisely, according to a private communication of Dirk van Dalen, " ... the decision of the Göttingen faculty to put Brouwer no 1 on the list for the chair was made on 30.10.1919".

example in his (1935), as "a remnant from that stage, at which this separation [between the formalism and metamathematical considerations] had not been made yet". The new ways are pursued in the lectures given during the winter term 1921/22.

4. FINITIST PROOF THEORY. These lectures contain for the first time the terms *finite Mathematik, transfinite Schlussweisen, Hilbertsche Beweistheorie*; their third part is entitled *Die Begründung der Widerspruchsfreiheit der Arithmetik durch die neue Hilbertsche Beweistheorie* (The founding of the consistency of arithmetic by the new Hilbertian proof theory). The clear separation of mathematical and metamathematical considerations allows Hilbert to address, finally, Poincaré's critique by distinguishing between *contentual metamathematical* and *formal mathematical* induction. This is most clearly presented in 1927, when Hilbert gave a second paper in Hamburg and claimed that Poincaré arrived at "his mistaken conviction by not distinguishing these two methods of induction, which are of entirely different kinds". Hilbert felt that "[u]nder these circumstances Poincaré had to reject my theory, which, incidentally, existed at that time only in its completely inadequate early stages".[13]

Weyl, no longer in the intuitionistic camp and no longer opposed to Hilbert's approach, responded to Hilbert's talk and turned the argument around justly claiming that "... Hilbert's proof theory shows Poincaré to have been exactly right on this point". After all, Hilbert had to be concerned not just with particular numerals, but "with an *arbitrary concretely given* numeral", and the contentual arguments of proof theory must "be carried out *in hypothetical generality*, on *any* proof, on *any* numeral". Such an understanding of quantification was explored already in the informal presentation of "finitist number theory" in the 1921/22 lectures. The interpretation is there no longer tied to a formal calculus that allows to establish free-variable statements, rather it assumes the "intuitive general concept of numeral" as part of the finitist standpoint.

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[13] p. 473 of (Hilbert 1927). How important this critique was can be seen from Weyl's remarks below, but also from the writings of others, for example, Skolem; see his papers (1922) and (1927). In the introduction to (Weyl 1927) in *From Frege to Gödel*, pp. 480-1, one finds a very thoughtful discussion of the underlying issues. -- In his 1922 paper (based on the Kopenhagen and Hamburg talks given in the Spring and Summer of 1921) Hilbert took still a different approach, consistent with the "strict finitist" view he had entertained, claiming that his metamathematical arguments did not involve mathematical induction and directly refuted Poincaré; however, in the contemporaneous (Bernays 1922) the distinction described above is made quite explicitly. -- In (Mancosu), 165-7, one can find a report on Becker's criticism of Hilbert's use of induction in metamathematical considerations.

In intuitive number theory the general sentences have a purely hypothetical sense. A sentence like a+b = b+a only means: given two numerals a, b, the additive composition of a with b yields the same numeral as the additive composition of b with a. There is no mention of the totality of all numbers. Furthermore, the existential sentences have in intuitive number theory only the meaning of partial-judgements, i.e., they are substatements of more precisely determined statements whose precise content, however, is inessential for many applications.

... thus, in general, a more detailed sentence complements in intuitive number theory an existential judgement; the sentence determines more precisely the content of that judgement. The existential claim here has sense only as a pointer to a search procedure which one possesses, but that ordinarily need not be elaborated, because it suffices generally to know that one has it.

This is exactly the understanding formulated in 1925 in *Über das Unendliche* and, most extensively, in 1934 in the first volume of *Grundlagen der Mathematik*; it is also strikingly similar to Weyl's viewpoint in (1921).[14] With this understanding of quantifiers the conclusion concerning the non-validity of the law of the excluded middle is obtained again. Hilbert writes (in the 1921/22 notes):

Thus we see that, for a strict foundation of mathematics, the usual inference methods of analysis must not be taken as logically trivial. Rather it is exactly the task for the foundational investigation to recognize, why it is that the application of transfinite inference methods as used in analysis and axiomatic set theory leads always to correct results.

As that recognition has to be obtained on the basis of finitist logic, Hilbert argues, we have to extend our considerations in a different direction in order to go beyond elementary number theory:

We have to extend the domain of objects to be considered; i.e., we have to apply our intuitive considerations also to figures that are not number signs. Thus we have good reason to distance ourselves from the earlier dominant principle according to which each theorem of pure mathematics is in the end a statement concerning integers. *This principle was viewed as expressing a fundamental methodological insight, but it has to be given up as a prejudice.*

This is a strong statement against a tradition that started with Dirichlet and includes such distinguished mathematicians as Weierstrass and Dedekind. But what is the new extended domain of objects, and what has to be preserved from the "fundamental methodological insight"? As to the domain of

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[14] Weyl's paper must have been known to Hilbert in 1921: in Hilbert's *Neubegründung der Mathematik* one finds the remark (on p. 160), "Wenn man von einer Krise spricht, so darf man jedenfalls nicht, wie es Weyl tut, von einer neuen Krise sprechen." This is obviously an allusion to the title of (Weyl 1921). According to (van Dalen 1995), p. 145, a draft of Weyl's paper was completed by May 1920, and a copy was sent to Brouwer. -- What is puzzling here is the circumstance that Weyl's views are, in some important respects (the understanding of quantifiers is one such point) close to the finitist standpoint; Weyl presents them as being different from Brouwer's, and Brouwer in turn recognizes immediately that Weyl is "in the restriction of the object of mathematics" even more radical than he himself; cf. (van Dalen 1995), p. 148 and p. 167. Why did it take the people in the Hilbert school such a long time to recognize that finitism was more restrictive than intuitionism? In a letter to Hilbert dated 25. X. 1925, Bernays mentions "a certain difference between the finitist standpoint and that of Brouwer"; but there is no elaboration of what this difference might be, and I don't know of any place where it is discussed by members of the Hilbert school before 1933. Indeed, in (Bernays 1930), the mathematical methods of finitism and intuitionism are viewed as co-extensional; it is only in the context of the Gödel-Gentzen reduction of classical to intuitionistic arithmetic that both Gödel and Gentzen point out that finitism is more restrictive than intuitionism; cf. (Gödel 1933), p. 294. This fact is then discussed in (Bernays 1934), p. 77; the significance of the result is described in (Bernays 1967).

objects, it is clear that the formulas and proofs from formal theories have to be included; as to the methodological requirements, Hilbert remarks:

... the figures we take as objects must be completely surveyable and only discrete determinations are to be considered for them. It is only under these conditions that our claims and considerations have the same reliability and evidence as in intuitive number theory.

From this *new standpoint* Hilbert exploits the formalizability of a fragment of number theory in full first order logic to formulate and prove its consistency. Here we finally close the gap to the published record -- with a fully developed programmatic perspective.

The dialectic of the developments that emerges from these lectures given between 1917 and 1922 is reflected in Bernays paper of 1922. Bernays' analysis brings out clearly the "Ansatzcharakter" of the proposed solution: in order to provide a rigorous foundation for arithmetic (that includes analysis and set theory) one proceeds axiomatically and starts out with the assumption of a system of objects satisfying certain structural conditions. However, in the assumption of such a system "lies something so-to-speak transcendental for mathematics, and the question arises, which principled position is to be taken [toward that assumption]". Bernays considers two "natural positions", positions that had been thoroughly explored. The first, attributed to Frege and Russell, attempts to prove the consistency of arithmetic by purely logical means; this attempt is judged to be a failure.

The second position is seen in counterpoint to the logical foundations of arithmetic: "As one does not succeed in establishing the logical necessity of the mathematical transcendental assumptions, one asks oneself, is it not possible simply to do without them." Thus one attempts a constructive foundation, replacing existential assumptions by construction postulates; that is the second position and is associated with Kronecker, Poincaré, Brouwer, and Weyl. The methodological restrictions to which this position leads are viewed as unsatisfactory, as one is forced "to give up the most successful, most elegant, and most proven methods only because one does not have a foundation for them from a particular standpoint".

Hilbert takes from these foundational positions, Bernays continues in his analysis, what is "positively fruitful": from the first, the strict formalization of mathematical reasoning; from the second, the emphasis on constructions. Hilbert does not want to give up the constructive tendency; on the contrary, he emphasizes it in the strongest possible terms. Finitist

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mathematics is viewed as part of an "Ansatz" to finding a principled position toward the transcendental assumptions:

Under this perspective[15] we are going to try, whether it is not possible, to give a foundation to these transcendental assumptions in such a way that only primitive intuitive knowledge is used.

The program is taken as a tool for an *alternative constructive* foundation of all of classical mathematics. The great advantage of Hilbert's method is judged to be this: "the problems and difficulties that present themselves in the foundations of mathematics can be transferred from the epistemological-philosophical to the properly mathematical domain." So Bernays, without great fanfare, gives an illuminating summary of about five years of quite intense work!

5. Remarks & issues. I find remarkable the free and open way in which Hilbert and Bernays joined, in the end, a number of different tendencies into a sharply focused program with a special mathematical and philosophical perspective. At first it seemed as if Hilbert's approach would yield results rather quickly and decisively: Ackermann's "proof" of the consistency of analysis was obtained already in 1923 and published in early 1924! However, difficulties emerged and culminated in the real obstacles presented by Gödel's Incompleteness Theorems.

The program has been transformed, quite in accord with the broad strategy underlying Hilbert's proposal, to a *general reductive* one; here one tries to give consistency proofs for strong classical theories relative to "appropriate constructive" ones. The first encouraging result was Gödel and Gentzen's reduction of classical elementary arithmetic PA to ist intuitionistic version HA. Even Gödel found the mathematical reductive program with its attendant philosophical one attractive in the thirties; his illuminating reflections, partly in an examination of Gentzen's first consistency proof for arithmetic, are presented in previously unpublished papers[16] that are now available in the third volume of his *Collected Works*. Foundationally inspired work in proof theory is being continued, weaving strong set theoretic and recursion theoretic strands into the metamathematical work.

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[15] of taking into account the tendency of the exact sciences to use as far as possible only the most primitive "Erkenntnismittel". That does not mean, as Bernays emphasizes, to deny any other, stronger form of intuitive evidence.

[16] I am thinking in particular of 1933A, 1938, and 1941.

The expanding development of proof theory is one effect of Hilbert's broad view on foundational problems and of his sharply articulated questions. Another effect is plainly visible in the rich and varied contributions that were given to us by Hilbert, Bernays, and other members of the Hilbert School (Ackermann, von Neumann, Gentzen, Schütte); finally, we have to consider also the stimulus his approach and questions provided to contemporaries outside the school (Herbrand, Gödel, Church, Turing and, much earlier already, Zermelo). Indeed, there is no foundational enterprise with a more profound and far-reaching effect on the emergence and development of mathematical logic. If we were open, it could have a similar effect on philosophical reflections on mathematical experience and help us gain a perspective that includes traditional concerns, but that allows us to ask questions transcending traditional boundaries.

Let me discuss briefly *one* such question. If we take the expansion of the domain of objects for finitist considerations seriously, we are dealing not just with numerals, but more generally with elements of inductively generated classes. (The generation is to be elementary and deterministic, in modern terminology.) A related point was already made by Poincaré, when he emphasized after discussing the principle of induction for natural numbers:

I did not mean to say, as has been supposed, that all mathematical reasonings can be reduced to an application of this principle. Examining these reasonings closely, we there should see applied many other analogous principles, presenting the same essential characteristics. In this category of principles, that of complete induction is only the simplest of all and this is why I have chosen it as a type." (p. 1025)

Gödel, following Poincaré and Hilbert, believed also that the method of complete induction has a "particularly high degree of evidence". But what is the nature of this evidence? In spite of important work that has been done for elementary number theory, this is still a significant question and should be addressed in greater generality. The assumption that work for elementary number theory covers all the bases, because of a simple effective Gödel numbering, prevents us from articulating the evidential features of inductively generated objects in a general way. That suggests two directions for interesting work.

First, there is ample room to improve our understanding of Hilbert's and Bernays' views on the matter. For example, I take it that Gödel's attempt to characterize the finitist standpoint in his 1958 paper is in conflict with their

views and with his own earlier informal description of the central features of finitist mathematics. At issue is whether the insights needed to carry out proofs concerning finitist objects spring purely from the combinatorial (spatiotemporal) properties of the sign combinations that represent them, or whether an element of "reflection" is needed, reflection that takes into account the uniform generation of the objects. The latter is explicitly affirmed in (Bernays 1930) and, by my lights, implicit in Hilbert's description of the "extra-logical concrete objects" that are needed to secure meaningful logical reasoning: such objects must not only be surveyable, but the fact that they follow each other, in particular, is immediately given intuitively together with the objects and cannot be further reduced.[17]

Second, and closer to contemporary proof theoretic investigations, there is an appropriate "generalization" of such considerations to classes that are obtained through generalized inductive definitions. The higher constructive number classes, already introduced by Brouwer and given a more precise definition by Church and Kleene, are a prime example. These i.d. classes have been used in reductions of impredicative subsystems of analysis; cf. (Buchholz e.a.). Coming back to my introductory remarks, let me emphasize that there is a mine for historical, logico-mathematical, and philosophical investigations: join in!

BIBLIOGRAPHY

Abrusci, Vito Michele
1981   "Proof", "Theory", and "Foundations" in Hilbert's mathematical work from 1885 to 1900; in: *Italian Studies in the Philosophy of Science*, M. L. Dalla Chiara (ed.), Dordrecht, 453-491.

Aspray, William and Philip Kitcher (eds.)
1988   *History and Philosophy of Modern Mathematics*, Minnesota Studies in the Philosophy of Science, Volume XI, Minneapolis.

Ackermann, Wilhelm
1924   Begründung des "tertium non datur" mittels der Hilbertschen Theorie der Widerspruchsfreiheit; Mathematische Annalen 93, 1-36.

Bernays, Paul
1918   Beiträge zur axiomatischen Behandlung des Logik-Kalküls; Habilitation, Göttingen.

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[17] That description is found in (Hilbert 1922) on pp. 162/3, but also later in (Hilbert 1925), p. 171, and (Hilbert 1927), p. 65.

1922	Über Hilberts Gedanken zur Grundlegung der Mathematik; Jahresberichte DMV 31, 10-19.
1926	Axiomatische Untersuchung des Aussagen-Kalküls der "Principia Mathematica"; Mathematische Zeitschrift 25, 305-320.
1930	Die Philosophie der Mathematik und die Hilbertsche Beweistheorie; in: (Bernays 1976), 17-61.
1935	Hilberts Untersuchungen über die Grundlagen der Arithmetik; in: (Hilbert 1935), 196-216.
1954	Zur Beurteilung der Situation in der beweistheoretischen Forschung; Révue internationale de philosophie 8, 9-13; Dicussion, 15-21.
1976	*Abhandlungen zur Philosophie der Mathematik*; Wissenschaftliche Buchgesellschaft, Darmstadt.

Brouwer, L.E.J.

1918	Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten; Erster Teil: Allgemeine Mengenlehre; KNAW 1e Sectie, deel XII, no. 5, 1-43.
1919	Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten; Zweiter Teil: Theorie der Punktmengen; KNAW 1e Sectie, deel XII, no. 7, 1-33.
1919A	Intuitionistische Mengenlehre; Jahresberichte DMV 28, 203-8; translated in (Mancosu).
1921	Besitzt jede reelle Zahl eine Dezimalbruchentwicklung?, Math. Annalen 83, 201-10.

Buchholz, Wilfried, Solomon Feferman, Wolfram Pohlers, and Wilfried Sieg

1981	*Iterated inductive defineitions and subsystems of analysis: recent prof-theoretical studies*; Lecture Notes in Mathematics, Springer Verlag.

Cantor, Georg

1897/9	Letters to Hilbert; in: (Purkert and Ilgauds).
1932	Gesammelte Abhandlungen mathematischen und philosophischen Inhalts; E. Zermelo (ed.); Berlin.

Dedekind, Richard

1888	Was sind und was sollen die Zahlen? in: (Dedekind 1932), 335-391.
1890	Letter to Keferstein; in: van Heijenoort, 98-103.
1932	Gesammelte mathematische Werke, Dritter Band; Fricke, Noether, and Ore (eds.), Braunschweig,1932.

Dreben, Burton and Jean van Heijenoort

1986	Introductory note to (Gödel 1929), 44-59.

Ewald, William B.

1996	*From Kant to Hilbert* - A source book in the foundations of mathematics; two volumes; Oxford University Press.

Feferman, Solomon

1988	Hilbert's Program relativized: proof-theoretical and foundational reductions, Journal of Symbolic Logic, 53 (2), 364-384.

Gödel, Kurt

1929	Über die Vollständigkeit des Logikkalküls; Dissertation, Vienna; reprinted and translated in: *Collected Works I*; 44-101.

| 1931 | Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I; reprinted and translated in: *Collected Works I*, 126-195. |

1931   Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I; reprinted and translated in: *Collected Works I*, 126-195.

1933   Zur intuitionistischen Arithmetik und Zahlentheorie; reprinted and translated in: *Collected Works I*, 282-295.

1933A   The present situation in the foundations of mathematics; in: *Collected Works III*, 36-53.

1938   Vortrag by Zilsel; in: *Collected Works III*, 62-113.

1941   In what sense is intuitionistic logic constructive?; in: *Collected Works III*, 186-200.

1958   Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes; in: *Collected Works II*, 217-251.

1986   *Collected Works I*; Oxford University Press, Oxford, New York.

1990   *Collected Works II* ; Oxford University Press, Oxford, New York.

1995   *Collected Works III*; Oxford University Press, Oxford, New York.

Goldfarb, Warren

1979   Logic in the twenties: the nature of the quantifier; Journal of Symbolic Logic 44 (3), 351-368.

Hallett, Michael

1989   Physicalism, reductionism & Hilbert; in: *Physicalism in Mathematics*, A.D. Irvine (ed.), Kluwer, Dordrecht, 183-257.

1995   Hilbert and Logic; in: *Québec Studies in the Philosophy of Science I*, M. Marion and R.S. Cohen (eds.) Kluwer, Dordrecht, 135-187.

Hilbert, David

1900   Über den Zahlbegriff; Jahresberichte DMV 8, 180-194.

1905   Über die Grundlagen der Logik und Arithmetik; translated in: van Heijenoort, 129-138.

*1917   Mengenlehre.

*1917/18   Prinzipien der Mathematik.

1918   Axiomatisches Denken; Mathematische Annalen 78, 405-415.

*1920 WS   Logik-Kalkül.

*1920 SS   Probleme der mathematischen Logik.

*1921/22   Grundlagen der Mathematik.

1922   Neubegründung der Mathematik; Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität 1, 157-177.

*1922/23   Logische Grundlagen der Mathematik.

1923   Die logischen Grundlagen der Mathematik; Mathematische Annalen 88, 151-165.

1925   Über das Unendliche; Mathematische Annalen 95, 1926, 161-190; translated in (van Heijenoort), 367-392.

1927   Die Grundlagen der Mathematik; Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität 6 (1/2), 1928, 65-85.

1935   *Gesammelte Abhandlungen*, 3. Band; Berlin.

N.B. The starred items are unpublished lecture notes that are contained in the Hilbert Nachlaß at the University of Göttingen. -- The translations (of texts from these sources) given in my paper are all mine.

Hilbert, David and Wilhelm Ackermann

1928   *Grundzüge der theoretischen Logik*; Springer, Berlin.

Hilbert, David and Paul Bernays

1934   *Grundlagen der Mathematik* , vol. 1; Springer, Berlin. (Second edition: 1968)

1939   ___, vol. 2. (Second edition 1970)

Mancosu, Paolo
1998   *From Brouwer to Hilbert* - The debate on the foundations of mathematics in the
                1920s; Oxford University Press.

Moore, Gregory
1988   The emergence of first-order logic; in: (Aspray and Kitcher), 93-135.

Peckhaus, Volker
1990   *Hilbertprogramm und Kritische Philosophie*; Vandenhoeck & Ruprecht.

Poincaré, Henri
1905   Les mathématiques et la logique; Revue de métaphysique et de morale, 13, 815-
                35; translated in (Ewald, vol. 2), 1021-38.
1906   Les mathématiques et la logique; Revue de métaphysique et de morale, 14, 17-
                34; translated in (Ewald, vol. 2), 1038-52.
1906A   Les mathématiques et la logique; Revue de métaphysique et de morale, 14, 294-
                317; translated in (Ewald, vol. 2), 1052-71.

Purkert, Walter and Hans Joachim Ilgauds
1987   *Georg Cantor*; Basel, Boston, Stuttgart.

Sieg, Wilfried
1984   Foundations for Analysis and Proof Theory; Synthese 60 (2), 159-200.
1988   Hilbert's Program Sixty Years Later; Journal of Symbolic Logic, 53(2), 338-348.
1990   Relative consistency and accessible domains; Synthese 84, 259-297.
1997   Aspects of mathematical experience; in: Philosophy of Mathematics Today,
                E. Agazzi and G. Darvas (eds.), Kluwer, 195-217.
1997A   Hilbert's Programs: 1917-1922; Technical Report, Carnegie Mellon University.

Skolem, Thoralf
1922   Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre;
                translated in (van Heijenoort), 290-301.
1929   Über die Grundlagendiskussionen in der Mathematik; in: *Den syvende
                skandinaviske matematikerkongress i Oslo 19-22 August 1929*, Broggers, Oslo,
                1930, 3-21.

Stein, Howard
1988   Logos, Logic, Logistike: Some philosophical remarks on the 19th century
                transformation of mathematics; in: (Aspray and Kitcher), 238-259.

van Dalen, Dirk
1995   Hermann Weyl's intuitionistic mathematics; The Bulletin of Symbolic Logic 1
                (2), 145-169.

van Heijenoort, Jean (ed.)
1967   *From Frege to Gödel, a source book in mathematical logic, 1879-1931*; Harvard
                University Press, Cambridge.

Weyl, Hermann
1910   Über die Definitionen der mathematischen Grundbegriffe; Mathematisch-
                naturwissenschaftliche Blätter 7, 93-95, 109-113.

1918   *Das Kontinuum*; Veit, Leipzig. (English translation by S. Pollard and T. Bole, Dover Publications, New York, 1987.)

1919   Der circulus vitiosus in der heutigen Begründung der Analysis; Jahresberichte DMV 28, 85-92.

1921   Über die neue Grundlagenkrise der Mathematik; Mathematische Zeitschrift 10, 39-79.

1927   Diskussionsbemerkungen zu dem zweiten Hilbertschen Vortrag über die Grundlagen der Mathematik; translated in (van Heijenoort), 480-484.