**APPM 5440: Applied Analysis I**

**Problem Set Two (Due Friday, September 11th)**

1. Let $(x_n)$ and $(y_n)$ be sequences of real numbers.

   (a) Show that
   $$\limsup_{n\to\infty} x_n = -\liminf_{n\to\infty}(-x_n).$$

   (b) Show that
   $$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$
   whenever the right-hand side is defined.

   (c) Show that
   $$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \leq \liminf_{n\to\infty}(x_n + y_n)$$
   whenever the left-hand side is defined.

2. Sup, Inf, Limit Interchanges:

   (a) Give an example of a double-indexed sequence $(x_{mn})$ for which
   $$\sup_m \lim_{n\to\infty} x_{mn} \neq \lim_{n\to\infty} \sup_m x_{mn}.$$

   (b) Show that, for any sequence of reals $(x_{mn})$, we can always exchange supremums:
   $$\sup_m \sup_n x_{mn} = \sup_n \sup_m x_{mn}.$$
   Can we always exchange infimums?

   (c) (H & N 1.10) Show that, for any sequence of real $(x_{mn})$,
   $$\limsup_{n\to\infty}\left(\inf_m x_{mn}\right) \leq \inf_m\left(\limsup_{n\to\infty} x_{mn}\right)$$
   and
   $$\sup_n\left(\liminf_{m\to\infty} x_{mn}\right) \leq \liminf_{m\to\infty}\left(\sup_n x_{mn}\right).$$

3. (H & N 1.12) Let $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Show that the composition
   $$h = g \circ f : X \to Z,$$
   defined by $h(x) = g(f(x))$, is also continuous.

4. Prove that uniformly continuous functions preserve Cauchy sequences. Thatis, if $f : X \to Y$ is uniformly continuous, and $(x_n)$ is a Cauchy sequence in $X$, show that $(f(x_n))$ is a Cauchy sequence in $Y$.

5. (H & N 1.8) So far in class, we have only talked about cluster points of sets. Let $(x_n)$ be a sequence of real numbers. A point $c$ on the extended real line ($c \in \mathbb{R} \cup \{\pm\infty\}$) is called a cluster point of the sequence $(x_n)$ if there is a convergent subsequence of $(x_n)$ with limit $c$.

   Let $C$ denote the set of cluster points of $(x_n)$. Prove that $C$ is a closed set (If we haven't gotten to the definition of a closed set yet, look it up!) and show that
   $$\limsup_{n\to\infty} x_n = \max C \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \min C.$$