Implication

There is another fundamental type of connectives between statements, that of *implication* or more properly *conditional statements.* In English these are statements of the form 'If p then q' or 'p implies q'.

Definition 1 *The compound statement $p \Rightarrow q$ ('If p then q') is defined by the following truth table:*

p	q	$p \Rightarrow q$
T	*T*	*T*
T	*F*	*F*
F	*T*	*T*
F	*F*	*T*

In an implicative statement, $p \Rightarrow q$, we call p the *premise* and q the *conclusion.*

The first two rows make perfect sense from our linguistic understanding of 'If p then q', but the second two rows are more problematical. What are we to do if p is false?

Note that we must do something, otherwise $p \Rightarrow q$ would not be a well defined *statement*, since it would not be defined as either true or false on all the possible inputs.

We make the convention that $p \Rightarrow q$ is always true if p is false.

The major reason for defining things this way is the following observation. made by Bertrand Russell (1872 - 1970).

> From a false premise it is possible to prove *any* conclusion.

The word *any* is very important here. It means literally anything, including things which are true.

It is a common mistake in proofs to assume something along the way which is not true, then proving the result is always possible. This is referred to as 'arguing from false premises'.

One problem is that in language we do not generally use implicative statements in which the premise is false, or in which the premise and and conclusion are unrelated.

We usually assume that an implicative statement implies a connection, this is not so in logic. In logic we can make no such presumption, who would enforce 'relatedness'? How would we define it?

When we wish to prove an implicative statement of the form $p \Rightarrow q$ we assume that p is true and show that q follows under this assumption. Since, with our definition, if p is false $p \Rightarrow q$ is true irrespective of the truth value of q, we only have to consider the case when p is true.

Converse, Inverse and Contrapositive

Given an implicative statement, $p \Rightarrow q$, we can define the following statements:

- The *contrapositive* is $\sim q \Rightarrow \sim p$.
- The *converse* is $q \Rightarrow p$.
- The *inverse* is $\sim p \Rightarrow \sim q$.

Theorem 2 *$p \Rightarrow q$ is logically equivalent to its contrapositive.*

Proof:

p	q	$p \Rightarrow q$	$\neg q$	$\neg p$	$\neg q \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Note that the converse is the contrapositive of the inverse.

A common method of proof is to in fact prove the contrapositive of an implicative statement. Thus, for example, if we wish to prove that

> For all p prime, if p divides n^2 then p divides n.

it is easier to prove the contrapositive:

> For all p prime, if p does not divide n then p does not divide n^2.

Only if and Biconditionals

Definition 3 *If p and q are statements:*

- p *only if* q *means 'If not* q *then not* p*' or equivalently 'If* p *then* q*'.*
  i.e. p *only if* q *means* $\Rightarrow q$*.*

- p *if* q *means 'If* q *then* p*' i.e.* $q \Rightarrow p$*.*

- *The* biconditional *'p if and only if q' is true when* p *and* q *have the same truth value and false otherwise. It is denoted* $p \Leftrightarrow q$*.*

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Notes

1. 'if and only if' is often abbreviated to iff.

2. In language it is common to say 'If p then q' when what we really mean is 'p if and only if q' - careful. See the remarks on page 26.

Theorem 4 $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

Proof:

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Thus $p \Leftrightarrow q$ means that both $p \Rightarrow q$ and its converse are true.

When we wish to prove biconditional statements we *must* prove each direction separately. Thus we first prove $p \Rightarrow q$ (if p is true then so is q) and then independently we prove $q \Rightarrow p$ (if q is true then so is p).

Necessary and Sufficient

Definition 5 *Given two statements p and q*

- *'p is a necessary condition for q' means $\sim p \Rightarrow \sim q$ or equivalently $q \Rightarrow p$.*

- *'p is a sufficient condition for q' means $p \Rightarrow q$.*

Notes

1. If p is a necessary condition for q this means that if q is true then so is p. However if p is true q may not be – there may be other things that must be true in order for q to happen.

2. If p is a sufficient condition for q then if p is true then q **must** happen as a consequence (all the conditions for q are fulfilled). However q may happen in some other way
   – so q True but p False is a possibility