Particle Accelerators

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European Spallation Source

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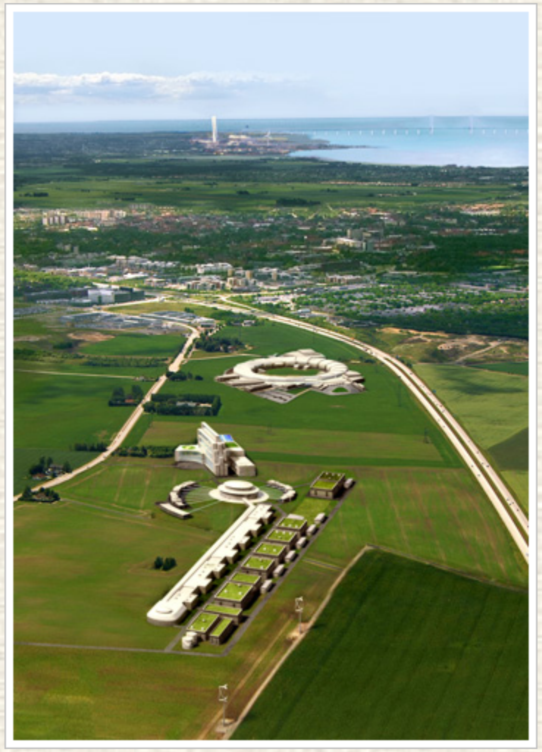
Contents

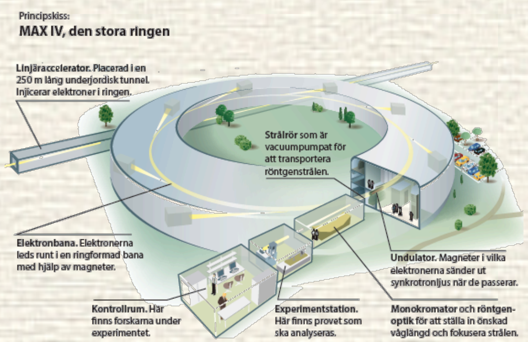
- Not all of accelerator technology: Not ion sources, diagnostics, controls, cryogenics, ...
- Not all about ESS or MAX IV: Not spallation targets, moderators, synchrotron radiation, ...

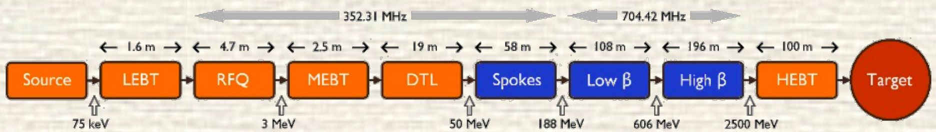
Accelerators, MAX IV, ESS

Category	Number in Use
High-energy accelerators (> 1 GeV)	~ 120
Synchrotron radiation sources	> 100
Medical radioisotope production	~ 200
Radiotherapy accelerators	> 7500
Research acc, including biomedical	~ 1000
Acc for industrial processing, research	~ 1500
Ion implanters, surface modification	> 7000
Sum 2004	> 17500

from Maciszewski and Scharf







Motion in Electric and Magnetic Fields

Non-relativistic

$\mathbf{F} = m\mathbf{a} = m\,\mathrm{d}\mathbf{v}/\mathrm{d}t$

$\mathbf{p} = m\mathbf{v}$

$\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$

$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$\mathrm{d}\mathbf{p}/\mathrm{d}t = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$E_{\mathrm{kin}} = mv^2/2$

Relativistic

$m = \gamma m_0 \neq const.$

$\mathbf{p} = m\mathbf{v} = \gamma m_0 \mathbf{v}$

$\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$

$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

$\mathrm{d}\mathbf{p}/\mathrm{d}t = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

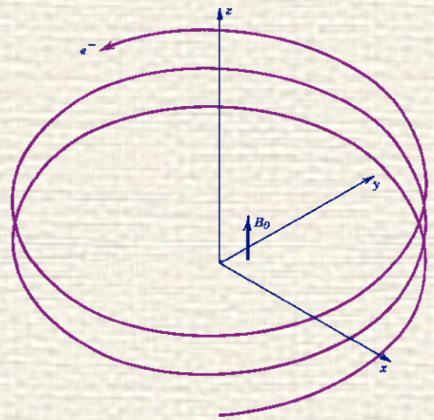
$E = mc^2 = \gamma m_0 c^2$

$E_{\mathrm{kin}} = (\gamma - 1) m_0 c^2$

$\gamma = 1/(1-\beta^2)^{-1/2}, \quad \beta = v/c$

e^- : $m_0c^2 = 511$ keV

p : $m_0c^2 = 938$ MeV



$r = mv/qB = p/qB$

Maxwell's Equations

Differential form

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial \mathbf{t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + 1/c^2 \, \partial \mathbf{E} / \partial \mathbf{t}$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

Integral form

$$\oint \mathbf{E} \cdot \mathrm{d}l = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}$$

Faraday's law

$$\oint \mathbf{B} \cdot \mathrm{d}l = \mu_0 \int \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot \mathrm{d}\mathbf{S}$$

Ampère's law

$$\int \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int \rho \, \mathrm{d}V$$

Gauss's law

$$\int \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

Potentials

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

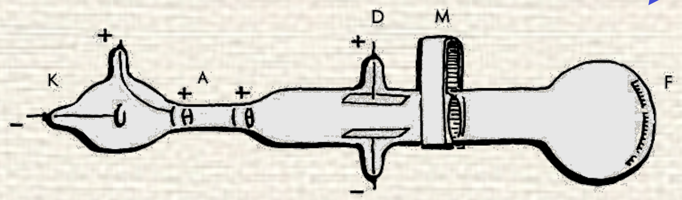
Waves in empty space

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

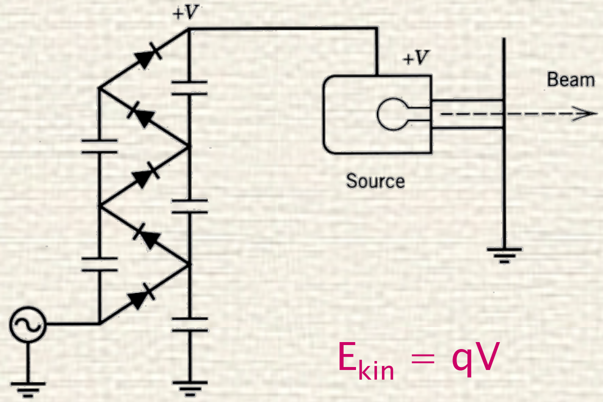
$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

Electrostatic Accelerators

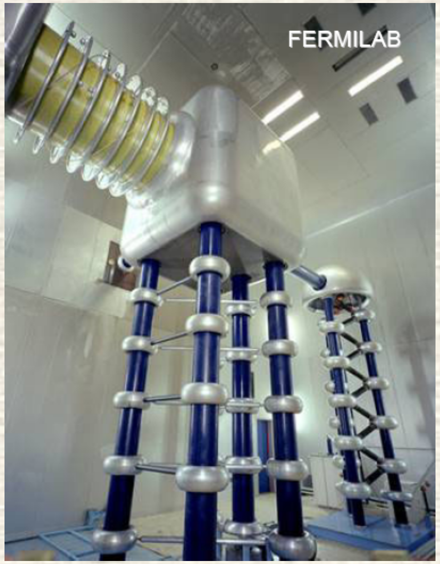
Cathode ray tube used by Thomson when discovering the electron



Teckningen visar det gasfyllda katodstrålerör som Thomson använde vid upptäckten av elektronen. I området mellan A och K alstras laddade partiklar. De negativt laddade partiklarna accelereras mot anoden A och passerar ett avböjningsarrangemang D och M i vilket partiklarna påverkas av elektriska och magnetiska krafter. Partikelstrålen får sedan träffa en fluorescerande skärm F. Läget på skärmen bestäms enbart av förhållandet mellan partikelns laddning och massa. Thomson fann att partiklarna träffade skärmen på samma ställe oberoende av vilken gasfyllning som användes, vilket antyder att alla atomer innehåller samma slags partikel. Denna partikel fick namnet elektron.



$E_{kin} = qV$

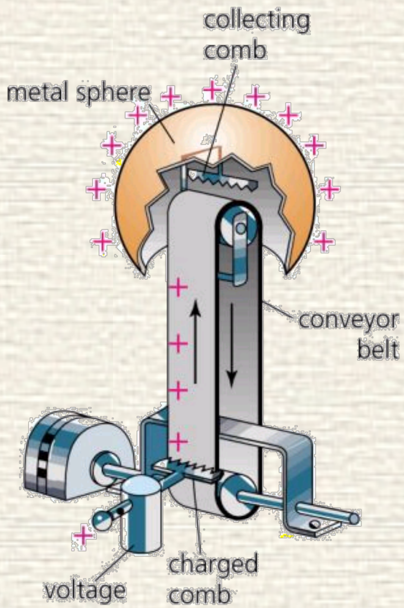


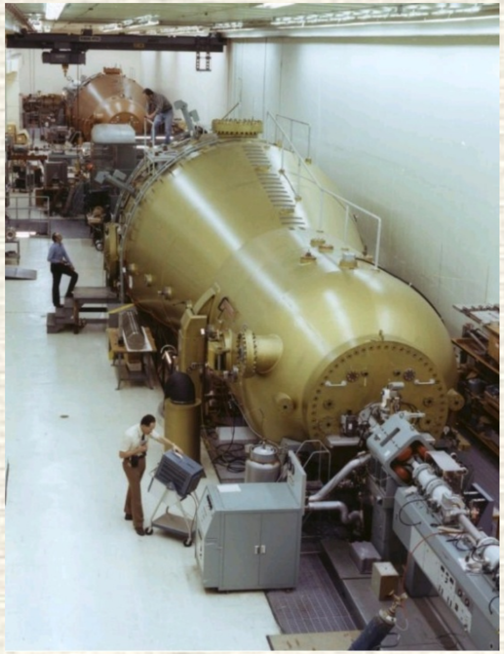
Cockcroft Walton generator, the first type of accelerator that “split the atom”

Cascade generator with capacitors and rectifier diodes can produce voltages up to several MV

Electrostatic Accelerators

Van de Graaff accelerator





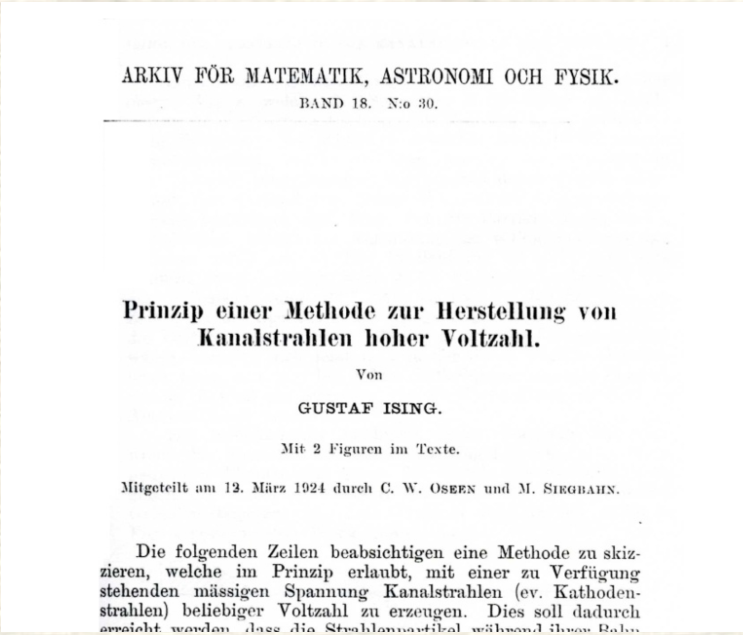
Brookhaven Nat'l Lab, 15 MV

Highest voltage with an electrostatic accelerator is 25.5 MV (Oak Ridge)

Gives 25.5(1+q) MeV in tandem configuration after stripping of negative ions in the high-voltage terminal

Accelerators with Time-Varying Fields

First idea by Gustaf Ising at Stockholms Högskola


ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK.
BAND 18. N:o 30.

Prinzip einer Methode zur Herstellung von Kanalstrahlen hoher Voltzahl.

Von

GUSTAF ISING.

Mit 2 Figuren im Texte.

Mitgeteilt am 12. März 1924 durch C. W. OSEEN und M. SIEGBAHN.

Die folgenden Zeilen beabsichtigen eine Methode zu skizzieren, welche im Prinzip erlaubt, mit einer zu Verfügung stehenden mässigen Spannung Kanalstrahlen (ev. Kathodenstrahlen) beliebiger Voltzahl zu erzeugen. Dies soll dadurch

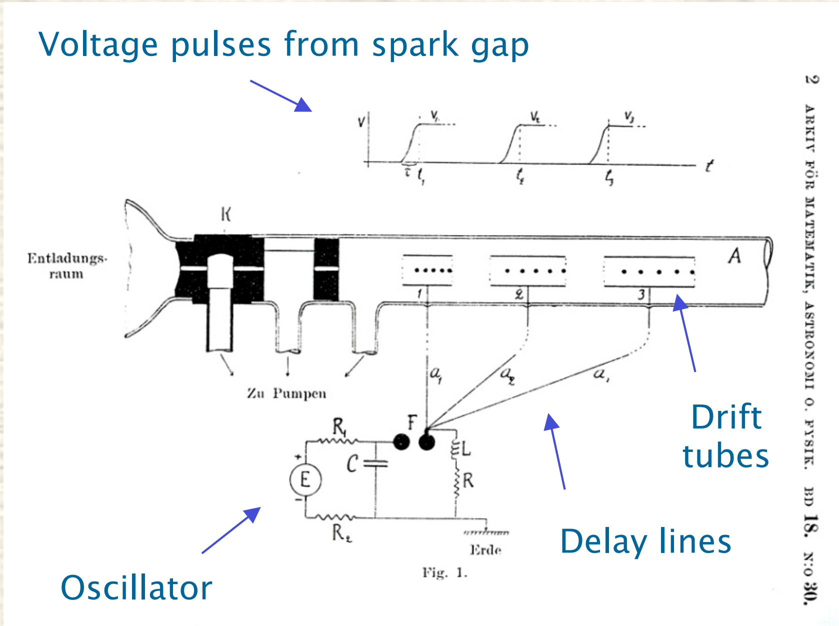
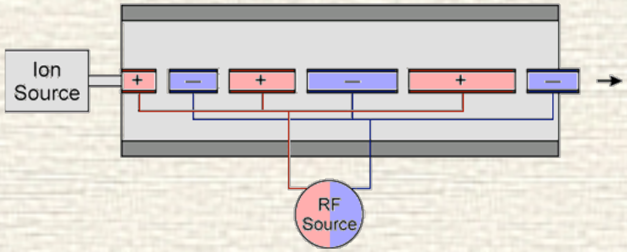


Fig. 1.

Drift-Tube Linacs

Wideröe



First working accelerator with AC fields, based on Ising's principle. Electrical fields only between drift tubes. Increasing drift tube length as energy increases. Low frequencies avoid power loss through radiation.

GSI, ions,
27 MHz

< 1999

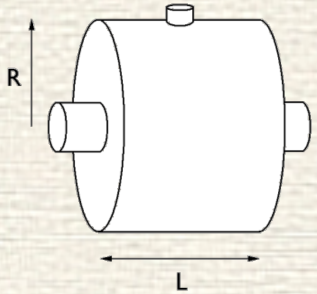
Alvarez

Fermilab DTL, protons, 201 MHz

Resonant electromagnetic cavity, fields everywhere inside the vacuum tank. Saves power, higher frequencies possible.

CERN Linac2, protons, 202 MHz

“Pill–Box” Cavity



Cylinder with conducting walls. Look for solution to the wave equation with only E_z and B_θ. Then E_z must be zero at r = R.

The wave equation for E_z in cylindrical coordinates is

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

There is a solution of the form

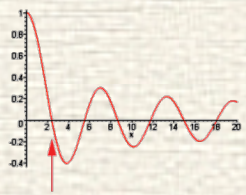
$$E_z(r,t) = E(r)e^{i\omega t}$$

E(r) must satisfy

$$E'' + \frac{E'}{r} + \left(\frac{\omega}{c}\right)^2 E = 0$$

The solution is

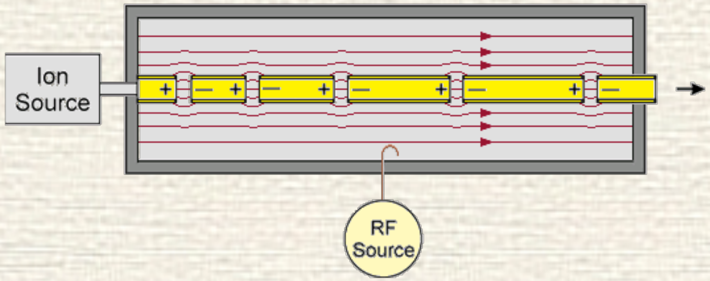
$$E(r) = E_0 J_0\left(\frac{\omega}{c}r\right)$$



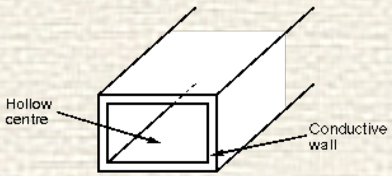
Requiring E = 0 at r = R gives

$$\frac{\omega}{c}R = 2.405$$

ESS has a DTL with f = 352 MHz, thus R = 0.33 m.



Waveguides



Electromagnetic fields propagating in waveguides (hollow conductors) can be used for acceleration of particles.

Looking for solutions to the wave equations proportional to exp[j(ωt-kz)], with $E_z = 0$ at the waveguide boundary, one finds waves with different symmetries or modes:

- TE_{mn} with transverse electric fields
- TM_{mn} with transverse magnetic fields
- (Hybrid modes)
- (TEM not possible)

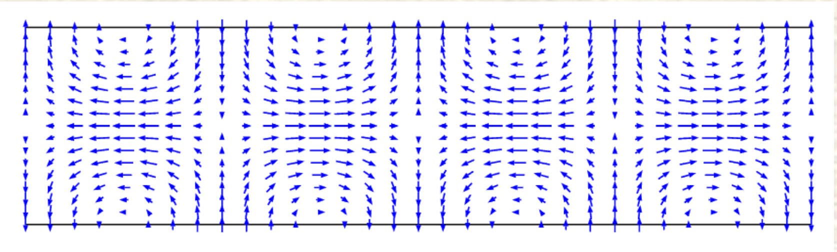
The simplest transverse magnetic mode (i.e. with longitudinal electric field that can accelerate particles) in cylindrical coordinates is TM_{01} where

$$E_z(r,z,t) = E_0 J_0(Kr) e^{j(\omega t - kz)}$$

$$E_r(r,z,t) = -j E_0 \frac{k}{K} J_1(Kr) e^{j(\omega t - kz)}$$

$$k^2 = \frac{\omega^2}{c^2} - K^2, \quad K = \frac{2.405}{R}$$

Electrical field of a TM_{01} wave with k = π/R, in a plane through the centre of the waveguide:



Travelling-Wave Linacs

DC

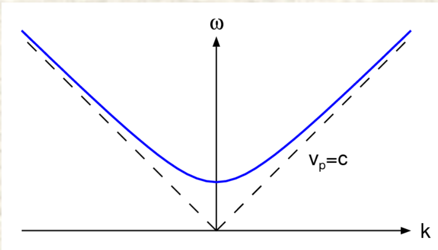
RF

In a travelling-wave linac, the particle speed must match the phase velocity of the wave:

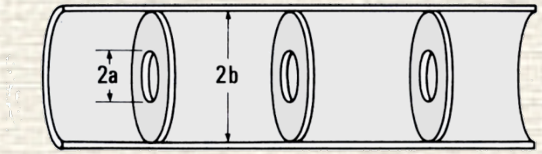
$$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1-(Kc/\omega)^2}} > c$$

$$(v_g = \frac{d\omega}{dk})$$

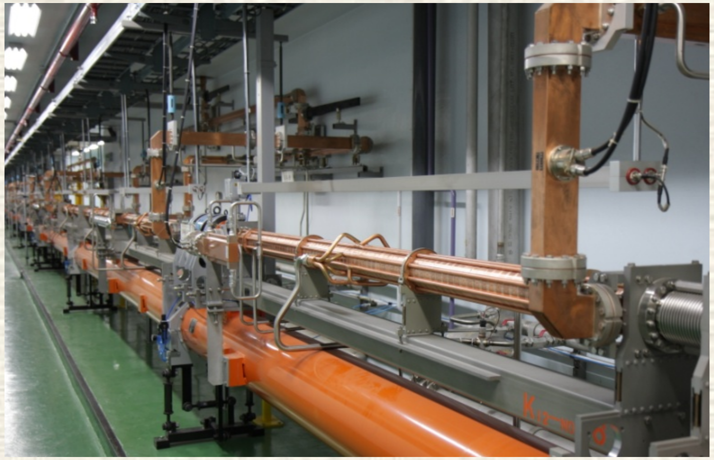
The speed of the wave depends on the frequency (dispersion) but is always > c.



The wave can be slowed down with apertures in the waveguide (disk-loaded cavities, suitable for electrons with v≈c).



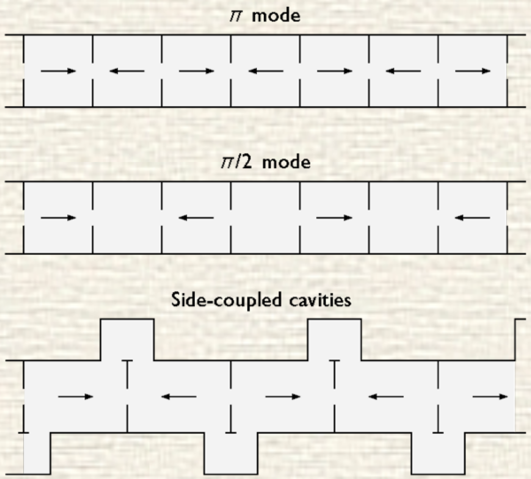
For an infinte such cavity, it can be shown that the field repeats from cell to cell, except for a phase shift.



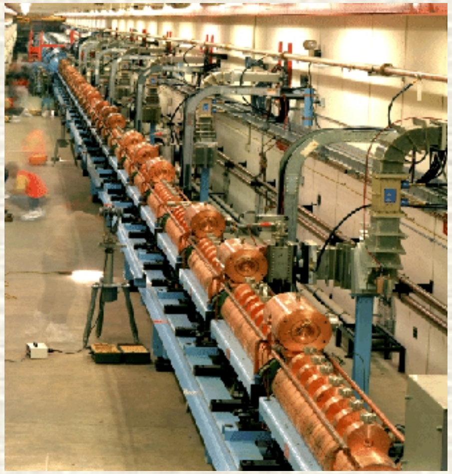
Pohang Light Source, 3 GHz

Standing-Wave Linacs

Standing-wave cavities (like pillboxes) can be coupled to longer structures of different geometries



A side-coupled cavity structure combines good mechanical and electromagnetic properties with high shunt impedance (see next slide).



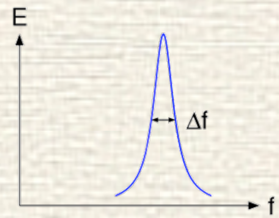
Fermilab side-coupled linac
protons, 805 MHz, 7.5 MV/m

Figures of Merit

The Q value of a resonator (of any kind) is the energy stored divided by energy loss per radian. For a pillbox, the stored energy can be calculated from the field, and also the current in the cavity walls which create the losses can be calculated, giving

$$Q = \frac{U}{P_\mathrm{d}/\omega} = \frac{\frac{1}{2}\varepsilon_0 \int E^2 \mathrm{d}V}{\frac{1}{2}\rho_\mathrm{s} \int J_\mathrm{s}^2 \mathrm{d}S \Big/ \omega} = \frac{2.405 \mu_0 c}{2\rho_\mathrm{s}(1 + R/L)}$$

using also Ampère's law to get $J_S = B_\theta/\mu_0$. For a copper cavity around 400 MHz, the Q value becomes ~10^4. The Q value also tells about the resonance width: $1/Q = \Delta f/f$.



The (effective) shunt impedance R_s is another figure of merit telling how efficiently rf power is transferred to the beam. It is the square of the energy gain per unit charge divided by the power dissipation:

$$R_\mathrm{s} = \frac{V^2}{P_\mathrm{d}} = \frac{Z_0^2}{\pi \rho_\mathrm{s}} \frac{L}{R} \frac{T^2}{(1 + R/L) J_1^2(2.405)}$$

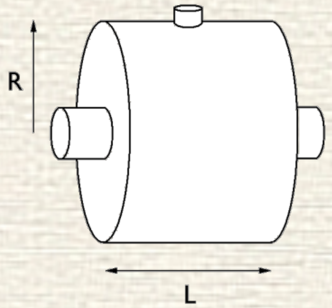
where the last expression holds for a pillbox, again, $Z_0 = 377\ \Omega$ and T is defined on next slide.

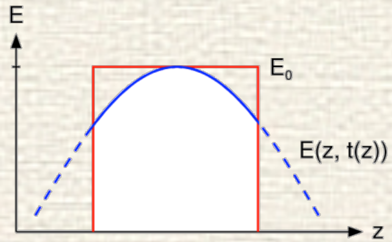
(Maybe, power to the beam divided by dissipated power would be more natural, but this depends on the RF amplitude.)

The shunt impedance per unit length is often used, since the shunt impedance itself depends on the accelerator length.

Finally, R_s/Q gives the acceleration efficiency per unit stored energy. It depends only on cavity geometry and frequency, not on the losses.

Transit Time Factor





$$T = \frac{E_0 \int_0^{L/2} \cos(\omega z / v) \mathrm{d}z}{E_0 L / 2} = \frac{\sin u}{u},$$

$$u = \frac{\omega L}{2v}$$

Setting T=0.9 (for example) and v≈c gives u=0.8 and L/R=0.65 for a pillbox cavity.

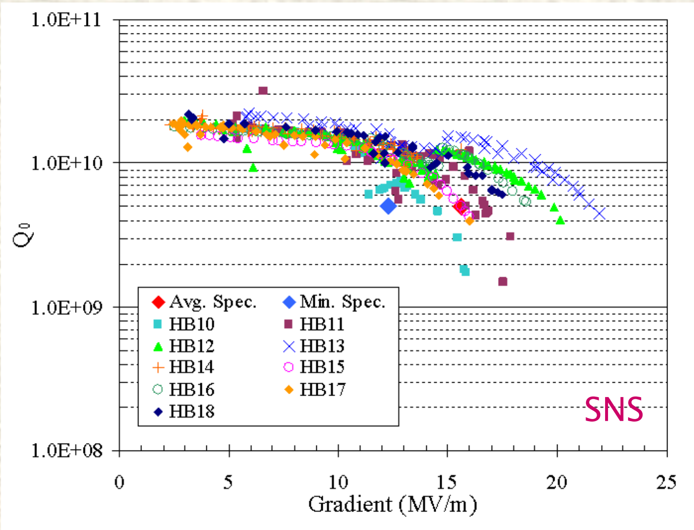
The transit time factor is introduced since the particles don't see the maximum field all the time. It is defined as the ratio between energy actually given to the particle and energy if the field were constant at its peak value.

CERN PS, 80 MHz (C. Plostinar)

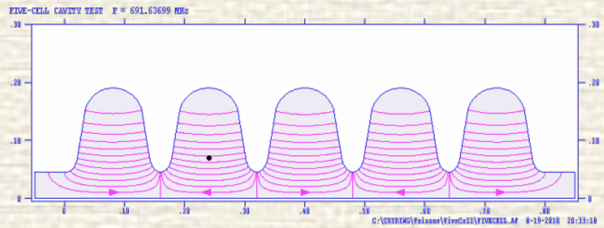
Superconducting RF

High beam currents and high duty factors give high losses in copper cavities which limits performance. Losses are orders of magnitude smaller in superconducting cavities, and they can also be given larger apertures. The cost and complexity now shifts to cryogenics.

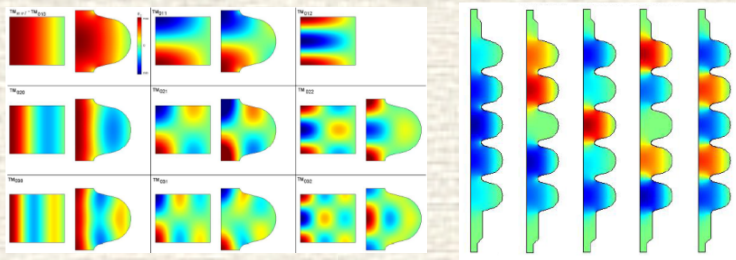
The Q value is an important figure of merit as well as maximum field strength.



XFEL 9-cell cavity



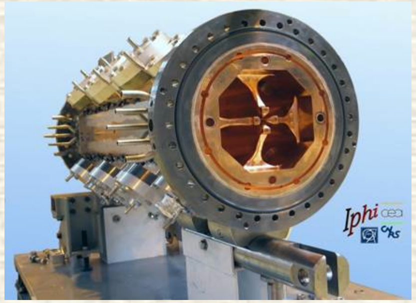
Field lines in ESS five-cell cavity (π mode)

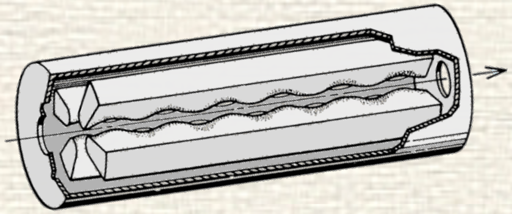


Other oscillation modes (Olivia Karlberg)

Non-Axisymmetric RF Structures

Radio Frequency Quadrupole (RFQ)

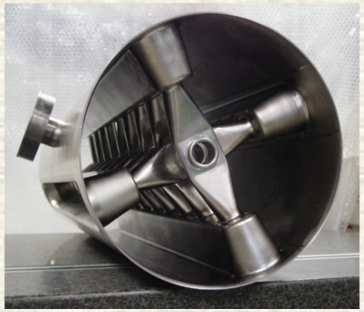




Bunches, focuses and accelerates a DC ion beam with near 100% transmission.

Interdigital H-mode (IH) DTL

Crossbar H-mode (CH) DTL



Single-spoke resonator

Double-spoke resonator

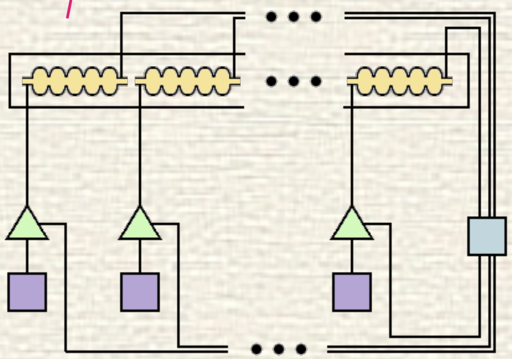
RF Power

SNS Klystron Gallery

SNS Cryomodule with four 6-cell elliptical cavities

SNS Low-level RF

SNS Modulator

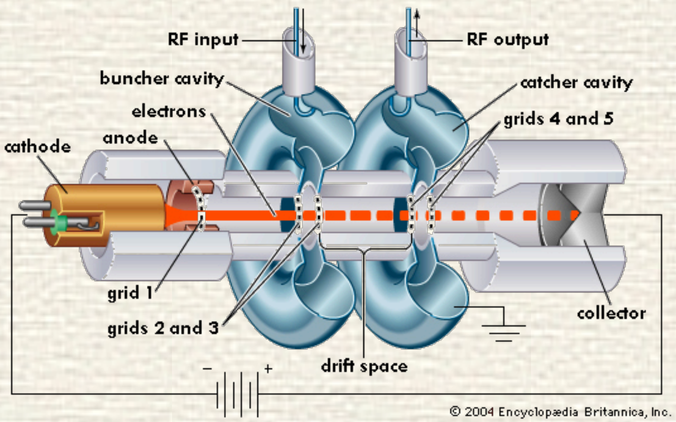


ESS, prel.	Quantity	SEK each
Cryomodules	41	10 M
Klystrons	203	2 M
Modulators	203	3 M

Sum, my rough estimate: 1425 MSEK

Microwave Sources

Klystron



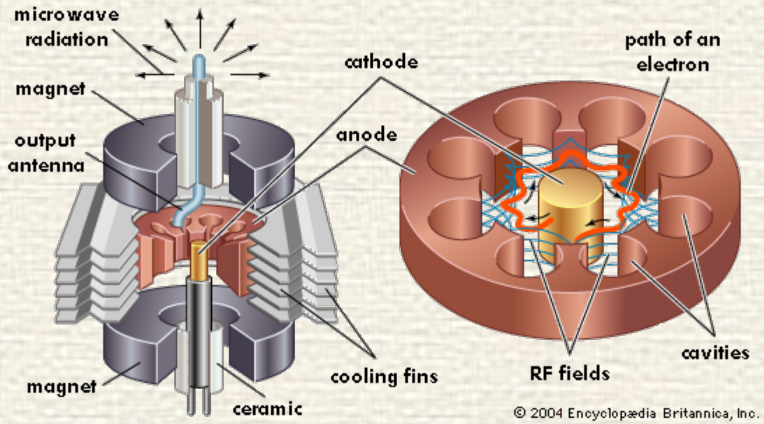
Small rf input gives small density perturbation in electron beam,
large density perturbation develops giving large rf output

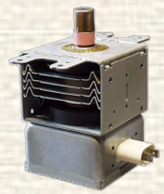
CPI VKP-7952C, 1.0 MW peak

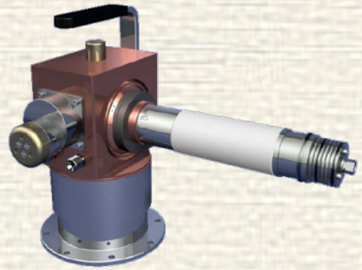
ESS (elliptical)

704 MHz
1.5 MW peak
5% duty factor
110 kV, 21 A

Magnetron





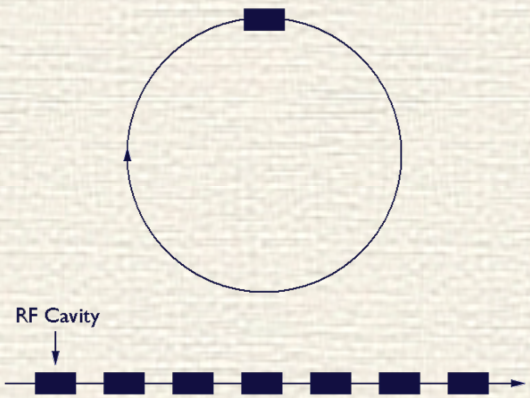


Transverse magnetic field makes electrons spiral from cathode to
anode, passing resonance cavities where output field is induced

Microwaves: 300 MHz - 300 GHz

Phase Stability

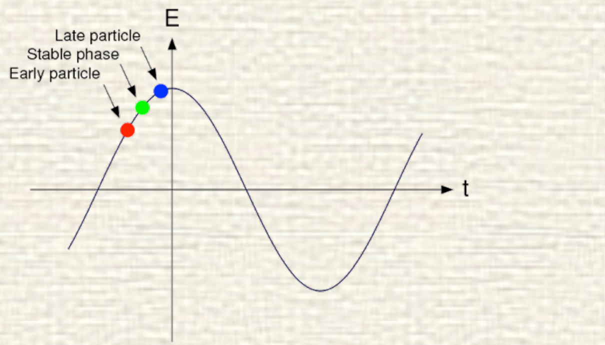
Consider particles that are accelerated in a periodic RF structure – in a circular or linear accelerator:



A particle keeps being accelerated if the RF frequency and/or cavity spacing is matched to the energy gain such that the particle enter each cavity at the right RF phase, in order to get the right energy gain, to match the increasing RF frequency or decreasing cavity spacing...

A particle which passes each gap at the same RF phase (assuming this is the design) is called the synchronous particle.

Phase stability ensures and is the consequence of the fact that that slow particles see a stronger electric field and get more accelerated and vice versa. Therefore, a certain velocity or energy spread can be accepted.



Slow particles come late, gain more energy.
Fast particles come early, gain less energy.
Synchronous particle stays at constant phase.
Only works for one sign of dE/dt.

Convention for RF is sine for circular machines, cosine for linacs. We think about linacs here, c.f. later...

Longitudinal Dynamics

A particle's phase with respect to the RF changes when it goes from one acceleration gap to the next according to

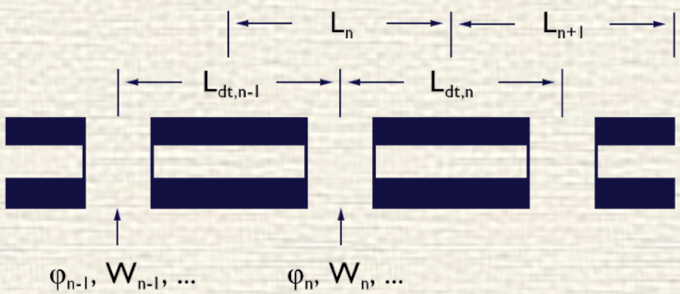
$$\varphi_n = \varphi_{n-1} + \omega \frac{L_{\mathrm{dt},n-1}}{\beta_{n-1} c}$$

The synchronous particle by definition has a speed given by

$$L_{\mathrm{dt},n-1} = \beta_{\mathrm{s},n-1} \lambda$$

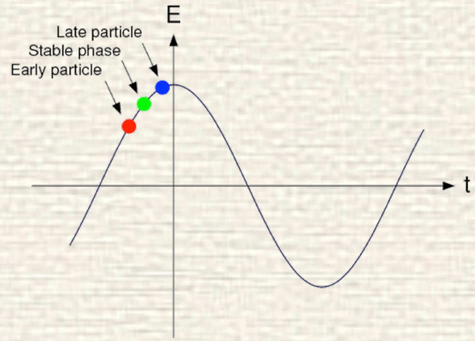
Convert to a cell from centre to centre of drift tube:

$$L_n = (\beta_{\mathrm{s},n-1} + \beta_{\mathrm{s},n}) \lambda / 2$$



Consider how the difference in phase between an arbitrary particle and the synchronous one changes from cell to cell:

$$\begin{aligned}
\Delta(\varphi - \varphi_{\mathrm{s}})_n &= \Delta\varphi_n - \Delta\varphi_{\mathrm{s},n} \\
&= (\varphi_n - \varphi_{n-1}) - (\varphi_{\mathrm{s},n} - \varphi_{\mathrm{s},n-1}) \\
&= \omega \frac{L_{\mathrm{dt},n-1}}{\beta_{n-1} c} - \omega \frac{L_{\mathrm{dt},n-1}}{\beta_{\mathrm{s},n-1} c} \\
&= \frac{\omega \beta_{\mathrm{s},n-1} \lambda}{c} \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{\mathrm{s},n-1}} \right) \\
&= 2\pi \beta_{\mathrm{s},n-1} \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{\mathrm{s},n-1}} \right).
\end{aligned}$$



Longitudinal Dynamics

Use

$$\frac{1}{\beta}-\frac{1}{\beta_s}=\frac{1}{\beta_s+\delta\beta}-\frac{1}{\beta_s}\approx-\frac{\delta\beta}{\beta_s^2},$$

$$\frac{\delta\beta}{\beta}=\frac{1}{\gamma^2\beta^2}\frac{\delta W}{W}=\frac{1}{mc^2\gamma^3\beta^2}\delta W$$

This results is a relation between change of phase difference and energy gain, and energy gain can be calculated from cavity voltage and phase:

$$\Delta(\varphi-\varphi_s)_n=\frac{2\pi(W_{n-1}-W_{s,n-1})}{mc^2\gamma_{s,n-1}^3\beta_{s,n-1}^2},$$

$$\begin{aligned}\Delta(W-W_s)_n&=\Delta W_n-\Delta W_{s,n}\\&=qE_0TL_n(\cos\varphi_n-\cos\varphi_{s,n})\end{aligned}$$

Here, q is the particle charge, E_0 the cavity peak voltage and T the transit time factor.

Now make the phase a continuous variable by changing differentials to derivatives and combine the two equations, assuming also that everything except W and phase are constants:

$$\begin{aligned}\frac{d^2(\varphi-\varphi_s)}{dn^2}&=\frac{2\pi qE_0TL(\cos\varphi-\cos\varphi_s)}{mc^2\gamma_s^3\beta_s^2}\\&\approx\frac{2\pi qE_0TL\sin\varphi_s}{mc^2\gamma_s^3\beta_s^2}(\varphi-\varphi_s)\end{aligned}$$

This shows that the phase performs harmonic oscillations about the synchronous phase with a frequency

$$Q_s=\left(-\frac{qE_0TL\sin\varphi_s}{2\pi mc^2\gamma_s^3\beta_s^2}\right)^{1/2}$$

expressed as number of oscillations per cavity spacing (the symbol Q not to be confused with quality factor).

Longitudinal Dynamics

A first integral is obtained by multiplying by $d(\varphi - \varphi_s)/dn$ and integrating over n. Assume $d\varphi_s/dn = 0$:

$$\int \frac{d^2(\varphi - \varphi_s)}{dn^2} \frac{d(\varphi - \varphi_s)}{dn} dn = \frac{2\pi q E_0 T L \sin\varphi_s}{mc^2 \gamma_s^3 \beta_s^2} \int (\cos\varphi - \cos\varphi_s) \frac{d(\varphi - \varphi_s)}{dn} dn$$

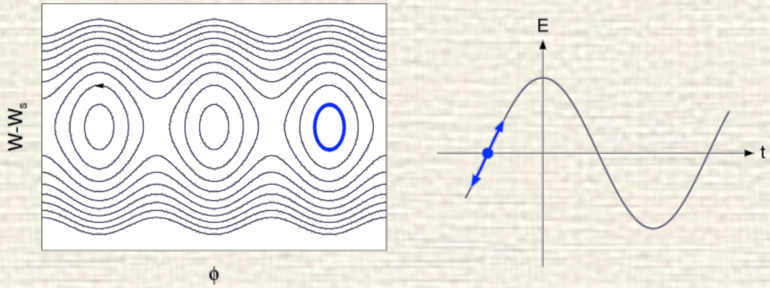
$$\frac{1}{2}\left(\frac{d(\varphi - \varphi_s)}{dn}\right)^2 + \frac{2\pi q E_0 T L \sin\varphi_s}{mc^2 \gamma_s^3 \beta_s^2} (\varphi \cos\varphi_s - \sin\varphi_s) = \text{const.}$$

Together with the expression for $\Delta(\varphi - \varphi_s)_n$ from the previous slide rewritten with derivatives we obtain

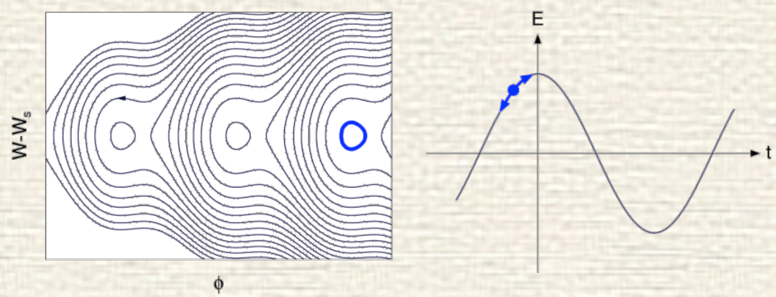
$$(W - W_s)^2 + \frac{2qE_0 T L mc^2 \gamma_s^3 \beta_s^2 \sin\varphi_s}{2\pi} (\varphi \cos\varphi_s - \sin\varphi) = \text{const.}$$

Energy difference as a function of phase can now be plotted for particles with different values of the integration constant, and for different synchronous phases:

No acceleration, $\varphi_s = -\pi/2$



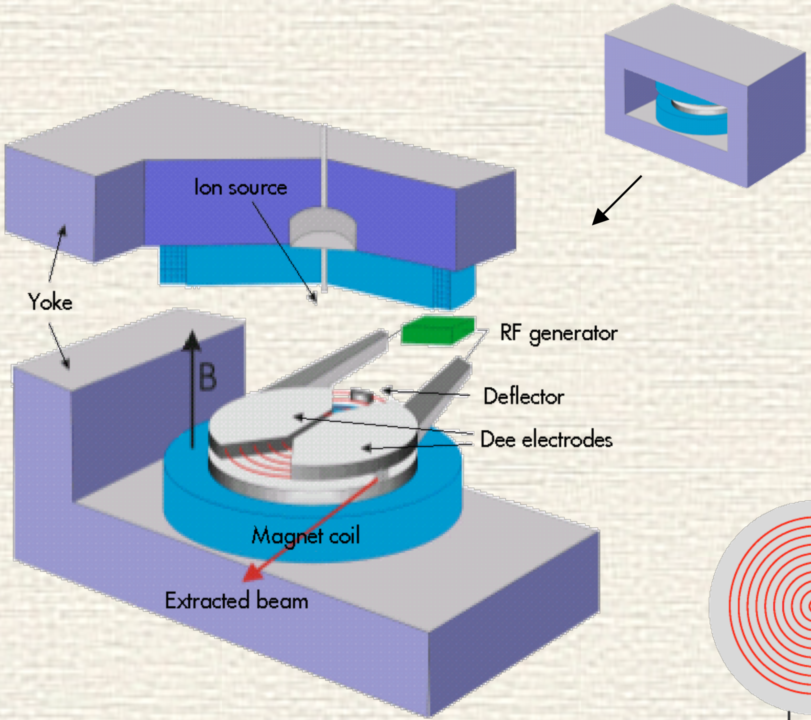
Acceleration, $\varphi_s = -\pi/5$

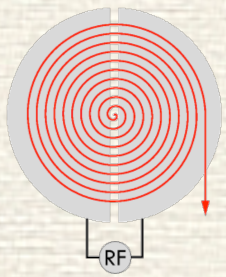


Classical Cyclotron

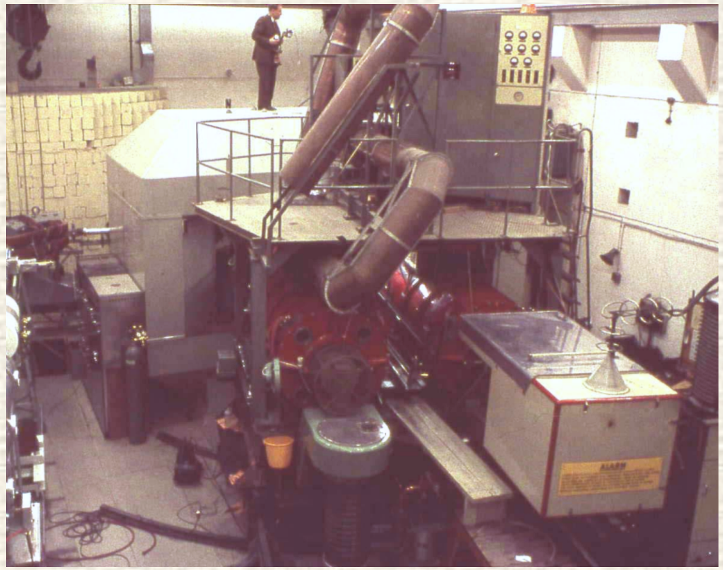
$\omega = qB/m = \text{const.}$

$r = p/qB$ increases



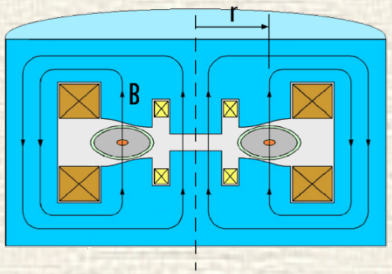


Nobel Institute
225-cm Cyclotron



Betatrons

In a betatron the particles move on a constant radius, and they are accelerated by the electrical field that is induced when the magnetic field is increased (rot E = −∂B/∂t)



Portable 6 MeV betatron for radiography

Constant radius:

$$r = \frac{p}{qB} = const. \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{qB}\right) = 0$$

$$\Rightarrow \quad \frac{\dot{p}}{qB} - \frac{p\dot{B}}{qB^2} = 0 \quad \Rightarrow \quad \frac{\dot{p}}{p} = \frac{\dot{B}}{B}$$

Induced field from change of flux inside r:

$$E = \frac{\dot{\Phi}}{2\pi r} \quad \Rightarrow \quad \dot{p} = \frac{q\dot{\Phi}}{2\pi r}$$

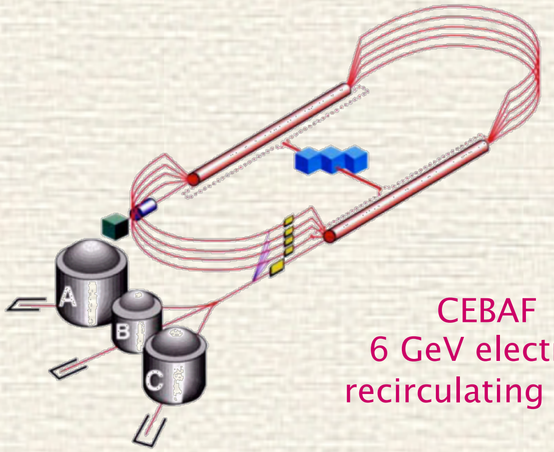
Together:

$$\dot{\Phi} = \frac{2\pi r\dot{p}}{q} = \frac{2\pi r p}{q}\frac{\dot{B}}{B} = 2\pi r^2 \dot{B}$$

Integrating and expressing the flux as πr^2 times the average field:

$$\Phi = 2\pi r^2 B + const. \quad \Rightarrow \quad \overline{B} = 2B + const.$$

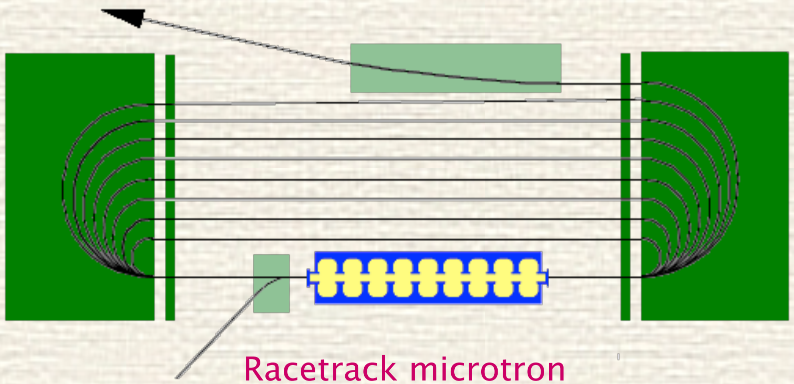
Microtrons, Recirculating Linacs



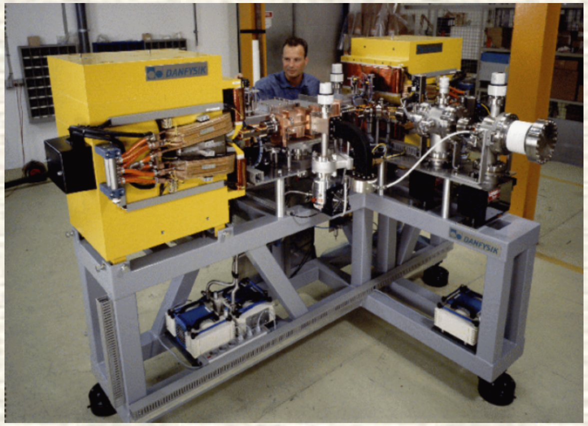
CEBAF
6 GeV electron
recirculating linac

Microtrons are used as electron accelerators for moderate energy and high reliability.

A racetrack microtron is similar to a recirculating linac, which is used for higher energies and has separate magnets for each orbit.



Racetrack microtron

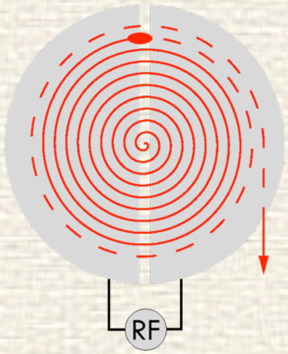


Danfysik 53 MeV microtron for ANKA

Synchro–Cyclotron

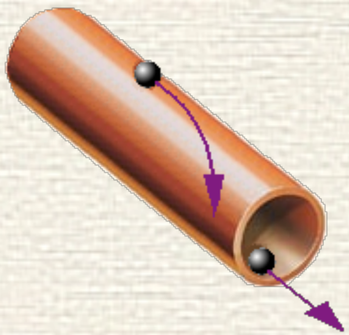
A classical cyclotron has a fixed frequency $\omega = qB/m$, but when particles become relativistic, $m = \gamma m_0$ increases and the particles get out of phase with the rf.

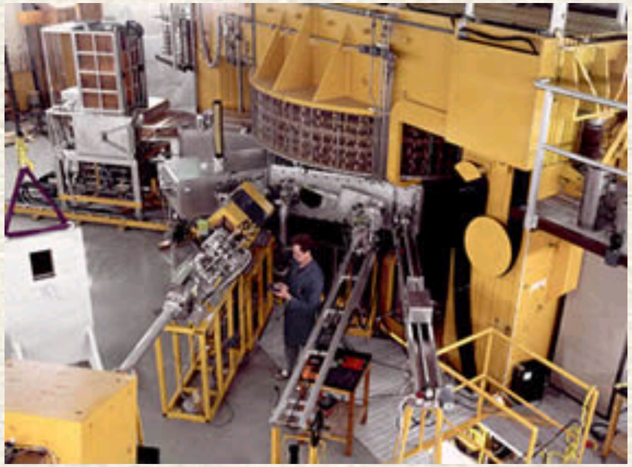
The magnetic field cannot increase with radius because particle orbits become unstable transversally. Instead ω must vary with time.



The frequency matches the relativistic mass only at one specific radius

The result is a pulsed beam and lower average current





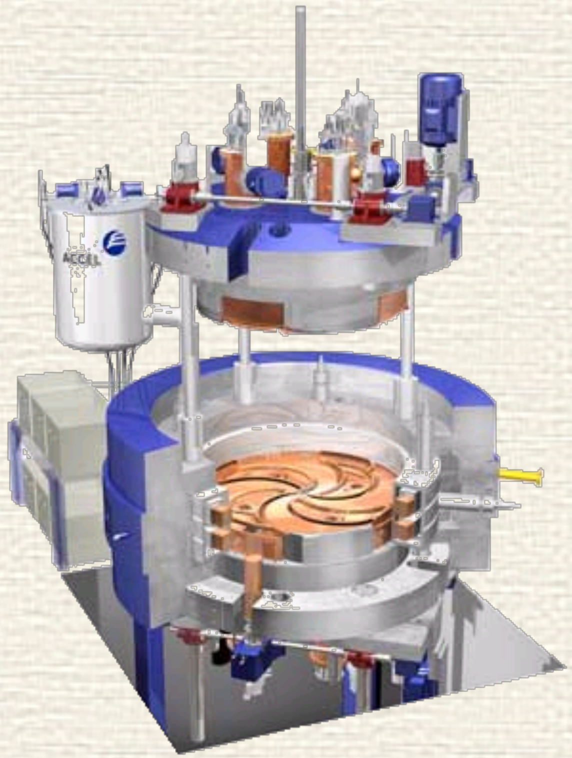
Gustaf Werner Cyclotron
(now mainly isochronous)

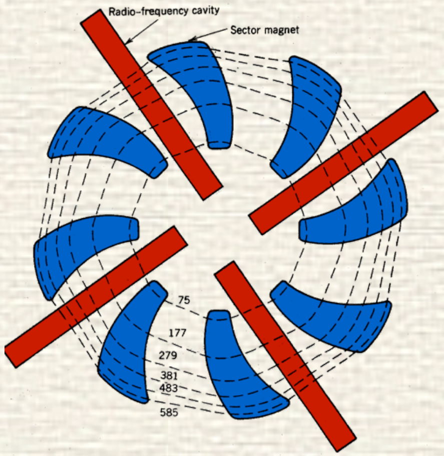
Isochronous Cyclotrons

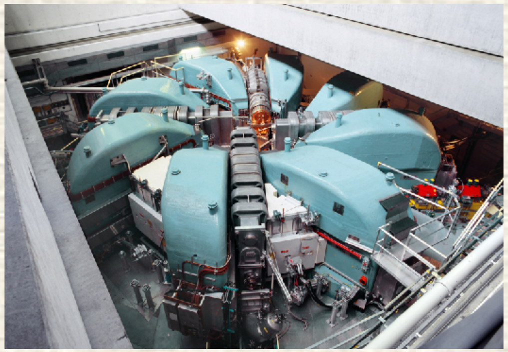
Later it was found out that azimuthally varying fields allow fixed frequency – same time for all orbits = isochronous – and orbit stability simultaneously, even for relativistic particles.

A ring geometry with separate magnets allows for more rf cavities, smaller gap and higher intensity.

Commercial superconducting 250 MeV proton cyclotron for cancer therapy







PSI ring cyclotron
> 1 MW protons

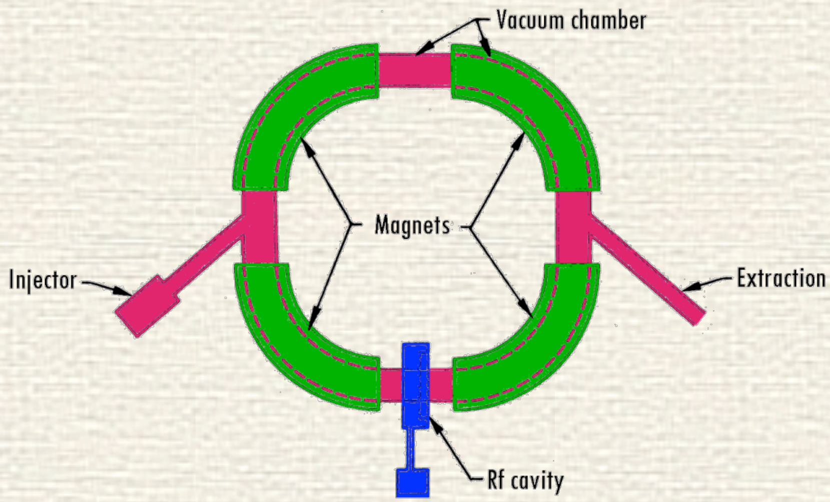
Synchrotrons

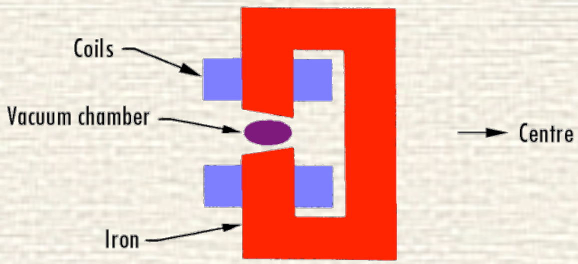
In a synchroton both magnetic field and frequency change with time ("synchronously") as the particles are accelerated, while the orbit radius remains constant.

This requires $B = p/qr$, $\omega = v/r$

Phase stability if slow particles see a higher accelerating field (?)

Transverse stability if $dB/dr < 0$





Diamond Light Source

Phase Stability

Does a higher-energy particle come around faster?

Use circumference C, orbit radius r, revolution time T, velocity v:

$$T = \frac{C}{v} \quad \Rightarrow \quad \frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = \frac{\Delta r}{r} - \frac{\Delta v}{v}$$

From definition $p = \gamma m_0 v$:

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

Bending radius in a magnet:

$$r = \frac{p}{qB} \quad \Rightarrow \quad p = qBr \quad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}r} = qB + qr\frac{\mathrm{d}B}{\mathrm{d}r}$$

$$\Rightarrow \quad \frac{\Delta p}{p} = \left(1 + \frac{r}{B}\frac{\mathrm{d}B}{\mathrm{d}r}\right)\frac{\Delta r}{r} \equiv \gamma_{\mathrm{tr}}^2 \frac{\Delta r}{r}$$

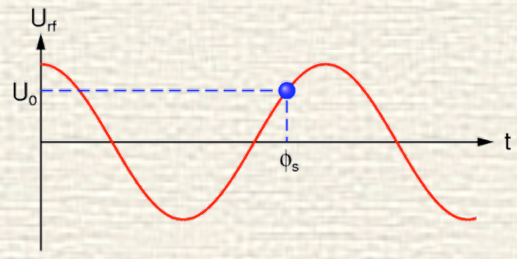
Define frequency slip factor, which can be positive or negative:

$$\eta = \frac{1}{\gamma_{\mathrm{tr}}^2} - \frac{1}{\gamma^2}$$

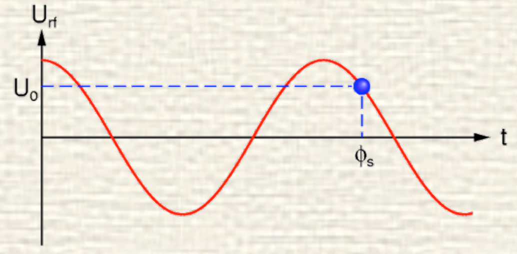
The result tells that the answer depends on the construction of the machine:

$$\frac{\Delta T}{T} = \eta \frac{\Delta p}{p}$$

Stable phase, η < 0



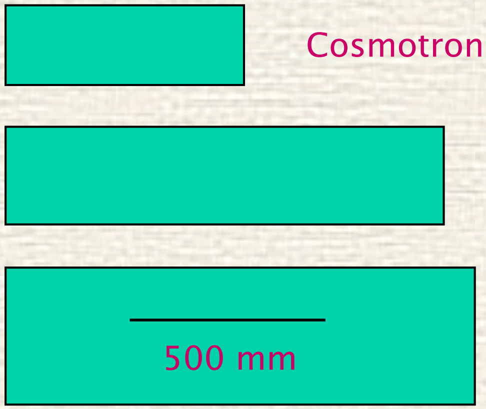
Stable phase, η > 0



What about η = 0?

Transverse Focusing

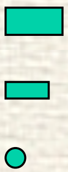
Size of magnet aperture



Cosmotron (3 GeV, 1952)

Bevatron
(6.2 GeV, 1954)

Synchrofasotron
(10 GeV, 1957)

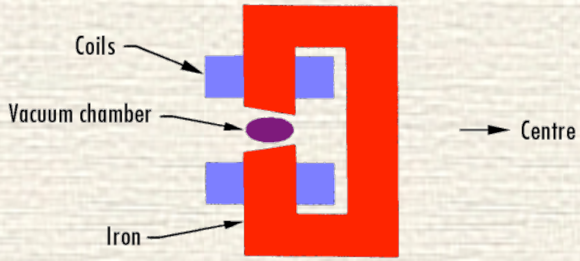


CERN PS (28 GeV, 1959)

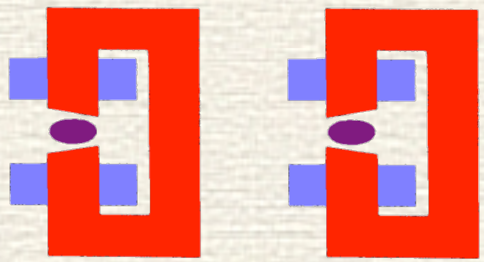
SPS (450 GeV, 1976)

LHC (7 TeV, 2008)

Weak focusing

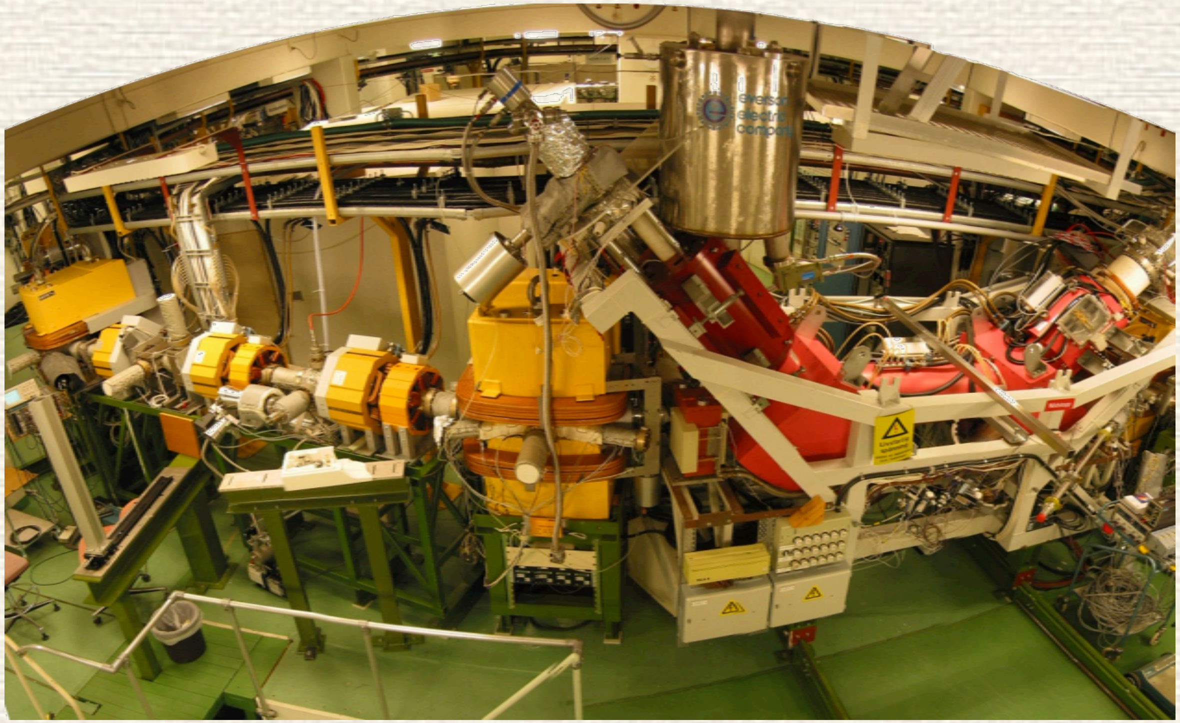


Alternating gradient



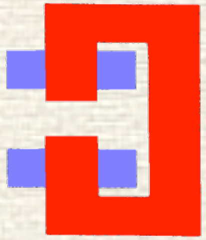
Separated Functions

CRYRING synchrotron at Manne Siegbahn Laboratory

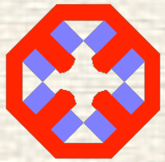


Different magnets for bending, focusing, ...

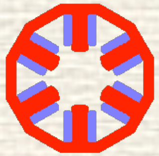
Dipole



Quadrupole

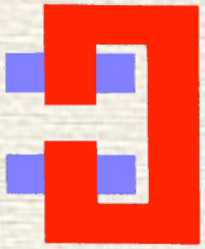


Sextupole

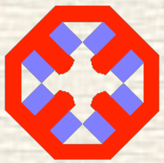


Multipole magnets

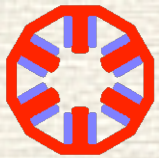
Dipole, B ~ const.



Quadrupole, B ~ r



Sextupole, B ~ r²



For magnetic fields in vacuum:

$$\nabla \times \mathbf{B} = 0, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Two dimensions:

$$\nabla^2 A_z(x,y) = 0$$

Separation of variables gives a solution, finite at the origin:

$$A_z = \sum_{n=1}^{\infty} r^n (a_n \cos n\varphi + b_n \sin n\varphi) = \mathrm{Re} \sum_{n=1}^{\infty} c_n (x+iy)^n$$

Take derivative and concentrate on "straight" as opposed to "skew" magnets (subscript zero means evalutated at the origin):

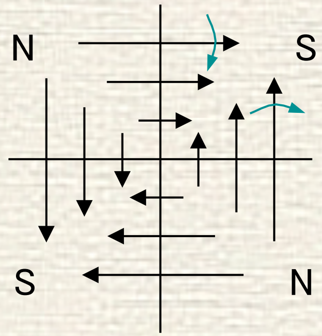
$$B_x = \frac{\partial A_z}{\partial y} = \frac{\partial B_{y,0}}{\partial x} y + \frac{\partial^2 B_{y,0}}{\partial x^2} xy + \ldots$$

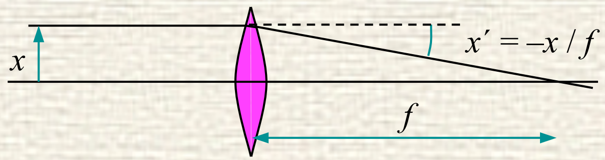
$$B_y = -\frac{\partial A_z}{\partial x} = B_{y,0} + \frac{\partial B_{y,0}}{\partial x} x + \frac{1}{2}\frac{\partial^2 B_{y,0}}{\partial x^2}(x^2 - y^2) + \ldots$$

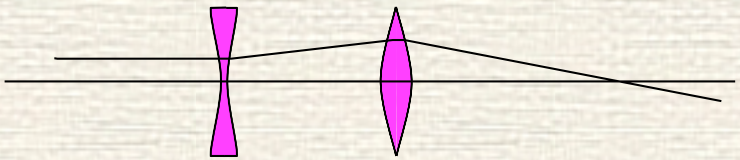
Dipole Quadrupole Sextupole

Lenses and Transfer Matrices

Quadrupole magnets focus in one plane and defocus in the other. They are linear in the sense that deflection is proportional to distance from optical axis, and parallel incoming rays are focused to a single point.







Transfer matrix for focusing lens with focal length f

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$$

Example:

$$\begin{pmatrix} 1 \\ -1/f \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\text{in}}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\text{in}}$$

Defocusing lens:

$$\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

Drift space of length L:

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Quadrupole doublet gives focusing in both planes

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} = \begin{pmatrix} 1+L/f & L \\ -L/f^2 & 1-L/f \end{pmatrix}$$

Transverse Stability

Using transfer matrices one can analyze if a periodic magnet lattice, like in a synchrotron, has stable particle orbits or not: Calculate a transfer matrix M for one entire period by multiplying individual matrices M_i for the N_p quadrupoles, drift spaces, edge focusing, etc. in the period:

$$M = M_1 M_2 \cdots M_{N_\mathrm{p}}$$

Introduce eigenvectors v_1 and v_2 with eigenvalues λ_1 and λ_2 defined by

$$M\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad i = 1,2$$

Any start vector can the be written as a linear combination of the eigenvectors

$$\begin{pmatrix} x \\ x' \end{pmatrix}_\mathrm{in} = A\mathbf{v}_1 + B\mathbf{v}_2$$

After a particle has made n periods through the lattice, its coordinates have become

$$\begin{pmatrix} x \\ x' \end{pmatrix}_\mathrm{out} = M^n \begin{pmatrix} x \\ x' \end{pmatrix}_\mathrm{in} = M^n (A\mathbf{v}_1 + B\mathbf{v}_2)$$
$$= A\lambda_1^n \mathbf{v}_1 + B\lambda_2^n \mathbf{v}_2$$

The motion is bounded if λ_i^n remain finite when $n \to \infty$. The question of transverse stability has now turned into a requirement on the eigenvalues of the transfer matrix for one magnet period.

Transfer matrices M_i for lenses, drift spaces, etc., have the determinant equal to 1, and since $|AB|=|A|\ |B|$, it follow that $|M|=1$. Furthermore, the product of the eigenvalues of a matrix equals its determinant. In two dimensions one thus has $\lambda_1 \lambda_2=1$ or

$$\lambda_1 = 1/\lambda_2$$

Transverse Stability

If one writes $\lambda_1 = e^{i\mu}$, it follow that $\lambda_2 = e^{-i\mu}$. The requirement that λ_i^n are finite can now be reformulated into stating that μ_i must be real.

Eigenvalues are found by solving the eigenvalue equation

$$|M - \lambda I| = 0$$

Writing

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

one has to solve

$$|M - \lambda I| = (ad - bc) - (a + d)\lambda + \lambda^2 = 0$$

We can use ad-bc = |M| and a+d is equal to the trace of M, Tr M. The eigenvalue equation then becomes

$$\begin{aligned} &1 - (\text{Tr } M)\lambda + \lambda^2 = 0 \\ \Rightarrow\quad &\lambda + 1/\lambda = \text{Tr } M \\ \Rightarrow\quad &e^{i\mu} + e^{-i\mu} = 2\cos\mu = \text{Tr } M \end{aligned}$$

The cosine of a real angle is always between -1 and 1, so there is now a recepie for putting toghether accelerator magnets:

Just make sure that the trace of the transfer matrix for an entire magnet period is between -2 and 2, and the particle will have stable orbits.

For a non-periodic lattice, like in a linear accelerator, this formalism is still useful when there are many, even though not infinitely many, identical lattice cells.

Transverse Beam Size

The size of the beam is an important quantity. With only dipoles and quadrupoles (linear focusing) and the orbit in one plane, the particle motion in each of the transverse directions will satisfy

$$x'' + K(s)x = 0$$

This is Hill's equation, and if K(s) is periodic, like in a synchrotron, Floquet's theorem says that there is a solution

$$x(s) = A\beta^{1/2}(s)\cos[\psi(s) + \delta]$$

where β(s) is a periodic function (for a stable machine) with the same period as K(s), and A and δ are constants of integration. The beta function and the phase satisfy

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$$

$$\psi(s) = \int \frac{\mathrm{d}s}{\beta(s)} + \text{const.}$$

where the last integration constant can be absorbed into δ, and only the phase advance from one point to another has physical meaning. The proofs of these statements are too long to be reproduced here.

The variables

$$\beta(s),\ \alpha(s) = -\frac{1}{2}\frac{\mathrm{d}\beta(s)}{\mathrm{d}s},\ \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

are known as Twiss parameters or Courant-Snyder parameters, and β determines the beam size together with A. Note that α, β, γ and the phase difference $\psi_1 - \psi_2$ between two points are properties of the machine while A and δ are properties of the particles.

Beta functions are useful also for linacs although they are not uniquely defined when the periodic boundary condition is missing.

Also important is the number of oscillations – betatron oscillations – that a particle performs in one turn, known as the tune or the Q value (called ν in the U.S.):

$$Q = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta(s)}$$

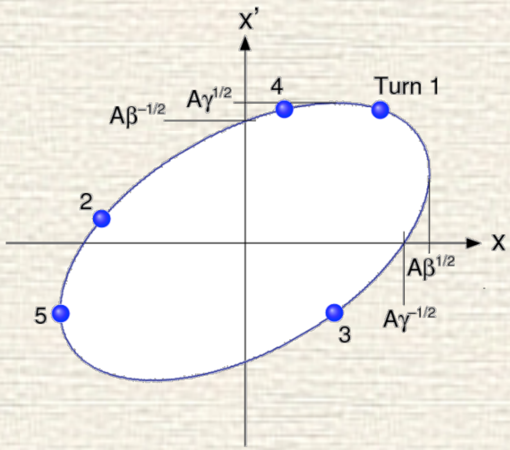
Emittance

Squaring and adding x(s) from last slide and the combination α(s)x(s)+ β(s)x'(s), one finds the relation

$$A^2 = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s)$$

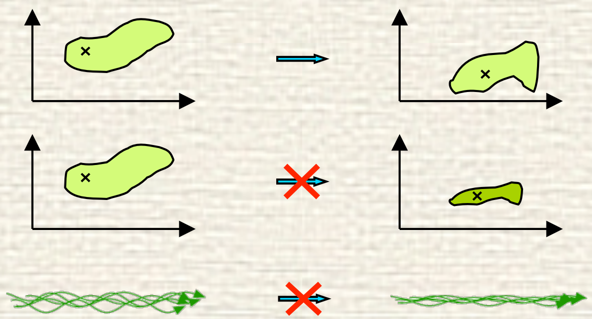
This is the equation of an ellipse in the x–x' plane for each position s in the accelerator, and πA^2 is the area of the ellipse.

Remembering that A is a constant of integration for the motion of each particle, this means that when a particle comes back to the same position in the ring, it appears on the same ellipse all the time.



All particles have ellipses of the same shape at the same s, and the area of the largest ellipse (or, e.g., that of a particle with the rms amplitude) defines the emittance ϵ of the beam. The emittance is clearly constant, at least as long as the beam is not accelerated. An accelerated beam gets an emittance inversely proportional to the momentum, i.e. to $1/(\beta\gamma)$. Therefore, a normalized emittance $\epsilon_N = \epsilon\beta\gamma$ is often used.

No fancy scheme of magnets or any other system of conservative forces can change the beam emittance as a consequence of Liouville's theorem. Beam cooling is about how to circumvent Liouville's theorem.



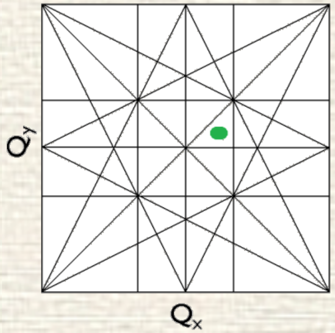
Resonances

The number of betatron oscillations per turn Q_x and Q_y should not be integers or rational numbers with small denominators, and neither should their sums and differences. In general on has to avoid

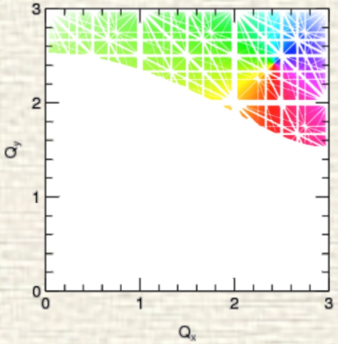
$$pQ_x + qQ_y = r$$

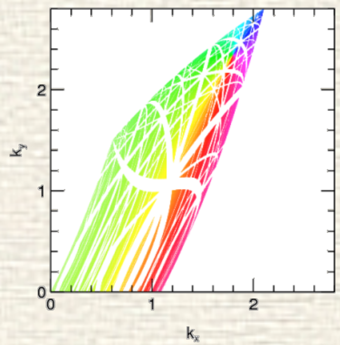
where p, q, and r are integers and p and q are small.

With, for instance, an integer Q a small magnet error (like a 10^{-5} too strong field in one magnet) gives an extra deflection of a particle at the same betatron phase at each turn, and the particle will soon hit the vacuum pipe. Like giving the swing a push at the right time all the time.



Resonances in a unit square for p, q ≤ 3 and working point between resonance lines.





Left: Stable Q values (below 3), from |Tr M| < 2, and resonance lines in CRYRING. Working point is where colours meet.

Right: Same but as a function of quadrupole strengths, which can be plotted in two dimensions since CRYRING has only two quadrupole families.

What Have We Learnt?

- Types of linear accelerators
- Standing waves, travelling waves
- Cavity types
- RF power
- Longitudinal stability
- Types of circular accelerators
- Transverse stability
- Multipole magnets, quadrupole lenses
- Transfer matrices
- Stability criterion
- Beta functions
- Emittance
- Resonances

Not enough?

- E. Wilson: An Introduction to Parcticle Accelerators (Oxford University Press, 2001)
- D.A. Edwards and M.J. Syphers: An Introduction to the Physics of High Energy Accelerators (John Wiley & Sons, 1993)