On Subrecursive Representability of Irrational Numbers

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We consider various ways to represent irrational numbers by subrecursive functions: via Cauchy sequences, Dedekind cuts, trace functions, several variants of sum approximations and continued fractions. Let $\mathcal{S}$ be a class of subrecursive functions. The set of irrational numbers that can be obtained with functions from $\mathcal{S}$ depends on the representation. We compare the sets obtained by the different representations.

A function $C : \mathbb{N} \to \mathbb{Q}$ is a *Cauchy sequence* for the real number α when $|\alpha - C(n)| < 1/2^n$. A function $D : \mathbb{Q} \to \{0, 1\}$ is a *Dedekind cut* of the real number α when $D(q) = 0$ iff $q < \alpha$. A function $T : \mathbb{Q} \to \mathbb{Q}$ is a *trace function* for the irrational number α when $|\alpha - q| > |\alpha - T(q)|$.

Any irrational number α can be written of the form $\alpha = a + \frac{1}{2^{k_0}} + \frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \ldots$ where $k_0, k_1, k_2, \ldots$ is a strictly monotone increasing sequence of natural numbers and a is an integer. Let $A : \mathbb{N} \to \mathbb{N}$ be a strictly monotone function. We will say that A is a *sum approximation from below* of the the real number α if there exists $a \in \mathbb{Z}$ such that $\alpha = a + \sum_{i=0}^{\infty} 1/2^{A(i)+1}$. Any real number can also be written as a difference between an integer and an infinite sum, and we will say that A is a *sum approximation from above* of the the real number α if there exists $a \in \mathbb{Z}$ such that $\alpha = a - \sum_{i=0}^{\infty} 1/2^{A(i)+1}$.

The sum approximations defined above are sum approximations in base 2. We will also consider *general sum approximations* (from above and below). A *general sum approximations* of α is a function that yields the sum approximation of α in any base.

An irrational number α can also be represented by a function $f : \mathbb{N} \to \mathbb{Z}$ where $f(n)$ yields the n^{th} element of the continued fraction $[a_0; a_1, a_2 \ldots]$ of α.

Let $\mathcal{P}_C$, $\mathcal{P}_D$ and $\mathcal{P}_{[]}$ denote the sets of irrationals that are representable, receptively, by primitive recursive Cauchy sequences, primitive recursive Dedekind cuts and primitive recursive continued fractions. Specker [3] proved $\mathcal{P}_D \subset \mathcal{P}_C$, and Lehman [2] proved $\mathcal{P}_{[]} \subset \mathcal{P}_D$ (strict inclusions). We will discuss a number on theorems on how trace functions and (general) sum approximation (from above and below) relate to Cauchy sequences, Dedekind cuts and continued fractions. Most of these theorems can be found in Kristiansen [1].

References

1. L. Kristiansen, *On subrecursive representability of irrational numbers.* Accepted for publication in Computability (the journal of CiE).
2. R. S. Lehman, On Primitive Recursive Real Numbers, *Fundamenta Mathematica* **49**(2) (1961), 105–118.
3. E. Specker, Nicht Konstruktiv Beweisbare Satze Der Analysis, *The Journal of Symbolic Logic* **14**(3) (1949), 145–158.