Let us be Antilogical: Anti-Classical Logic as a Logic

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1 Challenging Logicality

To be logical seems important, but what does it mean exactly? When are we logical and when are we illogical? Is it logical to think that $2+2 \neq 5$, that God does not exist, that it is impossible to go at a speed higher than light? It can be logical or not, it depends on what kind of thinking is behind, for which *reasons* we think so. We can also think logically or illogically the contrary, i.e. that $2+2=5$, that God exists, that it is possible to go at a speed higher than light.

It is important to make the distinction between logic as reasoning and logic as the theory of reasoning, a distinction that can be expressed as the difference *logic versus Logic* (for more details about this distinction, see [11]). René Descartes and Blaise Pascal were against a theory of logic such as syllogistic, but they were not against Logic, i.e. against being logical. Pascal wrote: "It is not Barbara and Baralipton that constitute reasoning. The mind must not be forced; artificial and constrained manners fill it with foolish presumption, through unnatural elevation and vain and ridiculous inflation, instead of solid and vigorous nutriment." [18]

For them being logical was to follow such basic principles such as:

- Never to accept anything for true which I did not clearly know to be such; that is to say, carefully to avoid precipitancy and prejudice, and to comprise nothing more in my judgment than what was presented to my mind so clearly and distinctly as to exclude all ground of doubt. (Descartes, [17])
- Not to employ in the definition of terms any words but such as are perfectly known or already explained. (Pascal, [18])

For them being logical is not to follow the rules of an artificial system such as syllogistic. But can we say the same nowadays after the Boolean revolution, the mathematization of logic, that has not only provided a more accurate description of reasoning but also transformed our way of reasoning, pushing the limits of rationality and developing artificial intelligence? There are thousands of systems of logic, from linear logic to erotetic logic, from alethic modal logic to turbo polar logic. It seems that it is possible to develop a logic for anything. Everything has its logic, everything is logical... There are even logics for reasoning with contradictions, the so-called "paraconsistent logics" (see e.g. [12]), which is absurd in the light of the principle of non-contradiction, which has been considered as the basis of logicality during many centuries.

To understand the nature of the logical in modern times, one has to inquire about the general nature of logical systems. This is what we will do here using anti-classical logic as a guide.

2 From Tarskian logical structures to anti-classical logic

Following the spirit of modern mathematics, we can say that a logic is a structure. But what kind of structure? The notion which has emerged is due to Alfred Tarski (his first ideas appear in [23], were reprinted in [13] with comments in [28]; see also [8]). It is a set with a consequence relation on it:

$$\mathcal{L} = \left\langle \mathbb{L}, \vdash_{\mathcal{L}} \right\rangle.$$

The consequence relation $\vdash_{\mathcal{L}}$ is a relation between a set of premises T, elements of $\mathbb{L}$, leading to a conclusion a, also an element of $\mathbb{L}$. Classical propositional logic, first-order classical logic and second-order classical logic can be seen as such logical structures. The many systems of intuitionistic, many-valued, paraconsistent and modal logics can also be seen in this way. However Tarski chose some axioms that exclude from this realm several kinds of logic.

The three Tarskian axioms are the following:

- **T1**: If $a \in T$, $T \vdash_{\mathcal{L}} a$.
- **T2**: If $T \vdash_{\mathcal{L}} a$ and $T \subseteq U$ then $U \vdash_{\mathcal{L}} a$.
- **T3**: If $T \vdash_{\mathcal{L}} a$ and $U, a \vdash_{\mathcal{L}} b$ then $T, U \vdash_{\mathcal{L}} b$.

These axioms say nothing explicitly about the logical operators: connectives, modalities, quantifiers, etc. So they are all welcome unless they indirectly contradict these axioms. These three axioms can be seen as respectively expressing reflexivity (**T1**), monotonicity (**T2**) and transitivity

(**T3**), some of the so-called "structural" properties of a logic. The most famous excluded logics from this framework are the non-monotonic logics promoted by John McCarthy and people working in artificial intelligence. Their argument for supporting such extravaganza is based on penguins and other empirical phenomena. Is it possible to find other creatures or objects that could serve as a basis for the rejection of reflexivity and/or transitivity? Yes, there are plenty of them. But rejection of Tarski's axioms can be based not only on empirical data but also on theoretical reasons.

There is a natural structure which obeys none of the Tarskian axioms, this is *anti-classical logic*. If we consider classical logic as the structure

$$\mathcal{K} = \left\langle \mathbb{K}, \models_{\mathcal{K}} \right\rangle,$$

anti-classical logic is then defined as

$$\overline{\mathcal{K}} = \left\langle \mathbb{K}; \models_{\overline{\mathcal{K}}} \right\rangle,$$

with the following definition of the anti-classical consequence relation:

$$T \models_{\overline{\mathcal{K}}} a \text{ iff } T \not\models_{\mathcal{K}} a.$$

Here are examples showing that anti-classical logic does not obey Tarskian axioms:

- **T1**: $e \not\models_{\overline{\mathcal{K}}} e$ (e being any proposition).
- **T2**: $d \models_{\overline{\mathcal{K}}} e$ and $\{d, e\} \not\models_{\overline{\mathcal{K}}} e$ (d, e atomic).
- **T3**: $d \models_{\overline{\mathcal{K}}} e$ and $e \models_{\overline{\mathcal{K}}} d$ but $d \not\models_{\overline{\mathcal{K}}} d$ (d, e atomic).

There are two issues:

- **I1**: Can we consider anti-classical logic as a logical structure despite the fact that it does not obey any Tarskian axioms?
- **I2**: Can we say that anti-classical logic describes a logical way of reasoning?

These two issues are intertwined. If we consider anti-classical logic as a structure of type $\mathcal{L} = \left\langle \mathbb{L}, \models_{\mathcal{L}} \right\rangle$, we can see $\models_{\mathcal{L}}$ as a road leading from some hypotheses T to a conclusion a, what should we specify for this road to be logical? Can be put no axiom on $\models_{\mathcal{L}}$? We can indeed argue in favor of axiomatic emptiness from some theoretical reasons.

Axiomatic emptiness has been promoted in the field of universal algebra by Garrett Birkhoff. He has developed a purely conceptual approach defining an algebra as a set with a family of operators obeying no axioms (see [15], [16]), by contrast with his predecessors Sylvester [22] and Whitehead [24], who were looking for some universal axioms for algebraic structures. For Birkhoff an abstract algebra is just a structure of type $\mathcal{A} = \langle \mathbb{A}, f_i \rangle$. This definition is enough to start working: in particular we can define the notion of subalgebra and morphism, they don't depend on any axioms. To have no axioms is no problem, it is in fact an advantage from a theoretical viewpoint, it allows to develop a smooth and universal theory. The same strategy can be applied to logical structures, this is the way to *universal logic* (cf. [2], [3], [7]).

From this perspective we can admit as logical structures any kind of structure of type $\mathcal{L} = \left\langle \mathbb{L}, \vdash_{\mathcal{L}} \right\rangle$. We have then two extreme cases, the logic in which nothing is a consequence of nothing, let us call it $\mathcal{L}_\emptyset$ (nickname: "zerologic") and the one in which everything is a consequence of everything, let us call it $\mathcal{L}_0$ (nickname: "cathologic"). If we consider a fixed domain $\mathbb{L}$, any logic on this domain is included in the cathologic on $\mathbb{L}$, includes the zerologic on $\mathbb{L}_\emptyset$, and it has an anti-logic which is also part of the network of logics defined on $\mathbb{L}$. We have a nicely structured class of logics. It is furthermore possible to use the square of opposition to describe the relations between logics and antilogics, as shown in [14] and [1].

If Eloise says that this is nonsense, Abelard will reply to her that this is not only nonsense, but general abstract nonsense. But we can be less Abelardian than Abelard arguing that anti-classical logic is a concrete logic. In fact it is possible to construct a semantics for anti-classical logic and also a proof-theoretical system, models and proofs being the two teats of modern logic, that sounds good and may produce a nice milky logic, on the basis of a completeness theorem establishing a link between these two sources of productions.

One of the godfathers of modern logic, the Polish logician Jan Łukasiewicz, has worked in this direction by developing refutation systems, i.e. proof-systems generating all formulas of a logic which are not logically valid (for recent works of the Polish School on this subject see [20], [21] [26]). If one considers that a logical argument is something than can be decomposed step by step, where each step can be justified by the application of a rule, then anti-classical logic proofs can be considered as logical arguments.

3 Substitution and replacement

There are two important metalogical features which are not valid in anti-classical logic. These are the so-called *substitution theorem* and *replacement* *theorem*. In modern logic there has been some ambiguity about the status of these two features, which is manifest in the expressions used to call them. Sometimes one of these expressions is used for the other, despite the fact that these are two quite different features — only apparently similar, false cognates — and moreover these properties are not necessarily theorems.

3.1 You shall not substitute

Substitutivity says that if we replace all the occurrences of an atomic formula by an arbitrary formula in a valid reasoning, then we still have a valid reasoning. In the Polish school Łoś and Suszko, pursuing the work of Tarski, have considered substitutivity as a fourth axiom they have added to the three Tarskian axioms, this led to the so-called *structural consequence relations* (cf. [19], reprinted in [13], with comments in [27]).

This property has a fundamental philosophical significance, which can be traced back to Aristotle, who was already using schematic letter to express it, and which makes sense within his hylemorphic views, i.e. the distinction between form and matter. In logic this means that logicality is formal in the sense that logical truth does not depend on the meaning but only on the form of the reasoning. This leads to a conception of logic based on logical forms.

However this property can be rejected for various reasons, in particular if one wants to take seriously in account meaning within the logical kingdom. People emphasizing meaning sometimes prefer to go outside of the logical realm, this was the position of Wittgenstein in his second period, who was rejecting logical systems as meaningless. Some other meaningful people apparently want to stay within the logical realm, saying they are doing "informal logic", but they are not dealing with logical systems. However it is in fact possible to deal with logical systems taking into account meaning, and a way to do it is to consider non-structural logics in which the substitution property does not hold.

From the perspective of axiomatic emptiness of universal logic [10], a logic structure does not necessarily obey the substitution axiom of Łoś and Suszko, a logic can be non-structural but still be a structure! And in this case we can still speak of *formal logic* if we consider that the form is located at a higher level of abstraction (cf. [9]).

One has to be careful: a logic in which the substitution property does not hold is not necessarily a logic of meaning. A good example is anti-classical logic which is a logical structure which is non-structural but which is not more meaningful than classical logic, the best known meaningless logic.

Let's see counterexamples of substitutivity in anti-classical logic. If we consider two atomic propositions such as e = *Snow is white* and d = *God is* *blue*, in anti-classical logic it is possible to deduce that *Snow is white* from *God is blue*, on the other hand it is not possible to deduce that *Snow is white* from the proposition f according to which *The sun is red and the sun is not red.* This is due to the fact that this reasoning is valid in classical logic, so we have here a typical failure of the substitution property. Symbolically, we have that $d \vdash_{\overline{\mathcal{K}}} e$, but $f \nvdash_{\overline{\mathcal{K}}} e$, since f is of the form $a \wedge \neg a$ and $a \wedge \neg a \vdash_{\mathcal{K}} e$.

Relevant logicians have also rejected the fact that from a contradiction anything follows, arguing that there are no meaningful connections in this case between the premises and the conclusion, and they have been paraconsistent for this reason. For them a way to try to catch meaning is to require that there are some atomic propositions in common between the premises and the conclusion. Anti-classical logic is not relevant, and it is also fully meaningless since, given two atomic propositions, one is a consequence of the other even if they have no common meaning.

In view of our above example, one may think that anti-classical logic is paraconsistent. This is a quite common deficient way of thinking, according to which a paraconsistent negation is any unary negation not obeying (one form) the ex falso sequitur quodlibet. But, as it has been stressed at length elsewhere (cf. [4] and [5]), a paraconsistent negation should also be defined positively, to be sure that we are talking about a kind of "negation", not an arbitrary unary connective. Thinking we are dealing with negation just because we are using the symbol "¬" is a symbol of the illusionism of symbolism. In the case of anti-classical logic, the use of the symbol "¬" is to keep in mind how the logic has been generated, but the connective denoted by "¬" in anti-classical logic is in no sense a negation.

This does not mean that there are no valid schemes of formula or consequence. For example, the proposition $e \wedge \neg e$ is valid in $\overline{\mathcal{K}}$ and so is any substitution of it. In anti-classical logic, among the tautologies, we have those which are individual and those which are schematic. This is an interesting distinction that can be used for any non structural logic. Examples of non schematic tautologies in $\overline{\mathcal{K}}$, besides atomic propositions, are formulas like $e \to (e \wedge d)$ or $(e \to d) \to (d \to e)$. In these two formulas if we substitute d for e then we have formulas which are not any more tautologies of $\overline{\mathcal{K}}$. A schematic tautology of $\overline{\mathcal{K}}$ is in fact an antilogy of K.

3.2 You shall not replace

The notion of logical equivalence can be defined for any consequence relation, independently of specific axioms. We say that two formulas a and b are logically equivalent in a logic $\mathcal{L}$ iff $a \vdash_{\mathcal{L}} b$ and $b \vdash_{\mathcal{L}} a$ This is abbreviated as $a \dashv\vdash_{\mathcal{L}} b$.

The replacement theorem says that if we replace an occurrence of a formula a by another formula equivalent to it b we preserve the consequence relation, in particular, if $a \underset{\mathcal{L}}{\dashv}\underset{\mathcal{L}}{\vdash} b$, then $c \underset{\mathcal{L}}{\dashv}\underset{\mathcal{L}}{\vdash} c(b/a)$, where c is a formula in which we have replaced a by b.

In anti-classical logic, the replacement theorem is not valid. We can consider the following counterexample: e and d are atomic formulas and they are logically equivalent since $e \underset{\overline{\mathcal{K}}}{\dashv}\underset{\overline{\mathcal{K}}}{\vdash} d$, but, if we replace the first occurrence of d by e in the formula $d \wedge \neg d$, then $d \wedge \neg d$ is not logically equivalent to $e \wedge \neg d$, because $d \wedge \neg d \underset{\overline{\mathcal{K}}}{\nvdash} e \wedge \neg d$.

The failure of the replacement theorem may appear as sheer logical nonsense. In the Polish school traditionally only logics in which this property is valid are considered as real logics. Following Wójcicki such logics are called self-extensional (see [25]), rightly emphasizing the significance of the replacement property. Now it seems that the reason why the Poles like this property is not because they are against intensional logic, but because it allows easy algebraization using Tarski-Lindenbaum methodology.

As it has been argued elsewhere, it seems logical to consider that a logic is intensional if it does not obey the replacement theorem, to qualify as intensional self-extensional modal logics such as $\mathcal{S}5$, $\mathcal{S}4$, etc. seems quite absurd (see [6]). But a logic which is not self-extensional is not necessarily intensional. Here again anti-classical logic is a good example: replacement is not valid, but it cannot really be considered as intensional. In classical logic and the standard modal logics, due to the replacement theorem, it is the same to say that a proposition is logically equivalent to itself and to say it is logically equivalent to a very different proposition. This extensional feature turns many fundamental mathematical theorems trivial. But anti-classical logic is not less trivial: a proposition is never logically equivalent to itself but can be equivalent to its negation.

4 Being highly logical

It is not seriously possible to argue that anti-classical logic is not logical because features such as substitution and replacement do not hold. On the contrary, one may argue that logics such as classical logic $\mathcal{K}$ or a modal logic $\mathcal{S}5$ are not logical because they have these features. By arguing in this direction one may just want to say that $\mathcal{K}$ and $\mathcal{S}5$ do not properly describe our natural way of thinking. We surely don't want to claim that anti-classical logic $\overline{\mathcal{K}}$ is a good description of the way we naturally think, but it is not much more absurd than $\mathcal{K}$ or $\mathcal{S}5$. The structure $\overline{\mathcal{K}}$ is a useful tool to study logicality in the same way that $\mathcal{K}$ and $\mathcal{S}5$ are useful tools to develop, extend and challenge our concept of logicality. Considering that being logical does not reduce to blindly follow some rules, some laws of thought, but also questioning logicality and creating new rules, we can claim that to develop anti-classical logic is to be highly logical.

BIBLIOGRAPHY

[1] H. Bensusan, A. Costa-Leite and E. Gonçalves de Souza, "Logics and their galaxies", forthcoming in *Logica Universalis*, vol. **8**, 2014.

[2] J.-Y. Béziau, "Universal Logic", in *Proceedings of the 8th International Colloquium — Logica'94*, edited by T. Childers and O. Majer, Czech Academy of Sciences, Prague, 2004, pp. 73–93.

[3] J.-Y. Béziau, *Recherches sur la Logique Universelle*, PhD Thesis, Department of Mathematics, University Denis Diderot, Paris, 1995.

[4] J.-Y. Béziau, "What is paraconsistent logic ?", in *Frontiers of paraconsistent logic*, Research Studies Press, Baldock, 2000, pp. 95–112.

[5] J.-Y. Béziau, "Are paraconsistent negations negations?", in *Paraconsistency: the logical way to the inconsistent*, edited by W. Carnielli et al, Marcel Dekker, New-York, 2002, pp. 465–486.

[6] J.-Y. Béziau, "The philosophical import of Polish logic", in *Methodology and philosophy of science at Warsaw University*, edited by M. Talasiewicz, Semper, Warsaw, 2002, pp. 109–124.

[7] J.-Y. Béziau (editor), *Logica Universalis — Towards a general theory of logic*, Birkhäuser, Basel, 2005, 2nd edition, 2007.

[8] J.-Y. Béziau, *Les axiomes de Tarski*, in *La logique en Pologne*, edited by M. Rebuschi and R. Pouivet, Vrin, Paris, 2006, pp. 135–149.

[9] J.-Y. Béziau, "13 Questions about universal logic", *Bulletin of the Section of Logic*, vol. **35**, 2006, pp. 133–150.

[10] J.-Y. Béziau, "What is a logic? Towards axiomatic emptiness", *Logical Investigations*, vol. **16**, 2010, pp. 272–279.

[11] J.-Y. Béziau, "Logic is not logic", *Abstracta*, **6** (2010), 73–102.

[12] J.-Y. Béziau, W.A. Carnielli and D.M. Gabbay (editors), *Handbook of Paraconsistency*, College Publication, London, 2007.

[13] J.-Y. Béziau (editor), *Universal logic: an Anthology — From Paul Hertz to Dov Gabbay*, Birkhäuser, Basel, 2012.

[14] J.-Y. Béziau, "The power of the hexagon", *Logica Universalis*, vol. **6**, 2012, pp. 1–43.

[15] G. Birkhoff, "Universal algebra", *Comptes Rendus du Premier Congrès Canadien de Mathématiques*, University of Toronto Press, Toronto, 1946, pp.310–326.

[16] G. Birkhoff, "Universal algebra", in *Selected papers on algebra and topology by Garrett Birkhoff*, edited by G.-C. Rota and J.S. Oliveira, Birkhäuser, Basel, 1987.

[17] R. Descartes, *Discours de la méthode (Discourse on the method)*, Leyde, 1637.

[18] B. Pascal, *De l'esprit géométrique et de lart de persuader (The art of persuasion)*, 1657.

[19] J. Łoś and R. Suszko, "Remarks on sentential logics", *Indigationes Mathematicae*, vol. **20**, 1958, pp. 177–183.

[20] T. Skura, "A refutation theory", *Logica Universalis*, vol. **3**, 2009, pp. 293–302.

[21] T. Skura, "On refutation rules", *Logica Universalis*, vol. **5**, 2011, pp. 249–254.

[22] J.J. Sylvester, "Lectures on the Principles of Universal Algebra", *American Journal of Mathematics*, vol. **6**, 1884, pp. 270–286.

[23] A. Tarski, "Remarques sur les notions fondamentales de la méthodologie des mathématiques", *Annales de la Société Polonaise de Mathématiques*, vol. **6**, 1928, pp. 270–271.

[24] A.N. Whitehead, 1898, *A treatise on universal algebra*, Cambridge University Press, Cambridge.

[25] R. Wójcicki, *Theory of logical calculi*, Kluwer, Dordrecht, 1988.

[26] U. Wybraniec-Skardowska and J. Waldmajer, "On Pairs of Dual Consequence Operations", *Logica Universalis*, vol. **5**, 2011, pp. 177–203.

[27] J. Zygmunt, "Structural consequence operations and logical matrices adequate for them", in *Universal logic: an Anthology — From Paul Hertz to Dov Gabbay*, edited by J.-Y. Béziau, Birkhäuser, Basel, 2012, pp. 163–176.
[28] J. Zygmunt, "Tarski's first published contribution to general metamathematics", in *Universal logic: an Anthology — From Paul Hertz to Dov Gabbay*, edited by J.-Y. Béziau, Birkhäuser, Basel, 2012, pp.59–67.