Relative stable pairs and a non-Calabi-Yau wall crossing

X smooth projective threefolds$/\mathbb{C}$

enumerative theory using sheaves on X:

smooth curves $C \hookrightarrow X \rightsquigarrow$ ideal sheaf $\mathcal{I}_C \hookrightarrow \mathcal{O}_X$.

$\mathcal{M}$ moduli of coh sheaves on X that contains ideal sheaves

$[\mathcal{M}]^{vir} \in A_{edim}(\mathcal{M})$.

intersections of $[\mathcal{M}]^{vir}$ with insertions classes on X

("counts" of curves on X with given incidence conditions)

1) ideal sheaves
2) stable pairs
3) relative stable pairs

1) ideal sheaves: Donaldson-Thomas theory

$\beta \in H_2(X,\mathbb{Z})$, $n \in \mathbb{Z}$

$\boxed{I_n(X,\beta)}$ moduli of ideal sheaves $\mathcal{I}_Z$,
$Z \hookrightarrow X$ subscheme with $[Z]=\beta \in H_2(X)$
$\chi(\mathcal{O}_Z)=n$.

same as the Hilbert sch of curves on X

Deformations: $\mathrm{Ext}^1_X(\mathcal{I}_Z,\mathcal{I}_Z)_0$
obstructions lie in $\mathrm{Ext}^2_X(\mathcal{I}_Z,\mathcal{I}_Z)_0$

defs for fixed det

trace-free parts of Ext gps

$$\underline{\underline{\text{edim}}} = \dim \mathrm{Ext}^1_X(\mathcal{I}_Z,\mathcal{I}_Z)_0 - \dim \mathrm{Ext}^2_X(\mathcal{I}_Z,\mathcal{I}_Z)_0$$
$$= \beta . c_1(X).$$

Rmk. In general, $I_n(X,\beta)$ does not have this expected dimension.

However, there is $[I_n(X,\beta)]^{vir} \in A_{\text{edim}}(I_n(X,\beta))$.

Donaldson-Thomas invariants

$\mathbb{I}$ univ ideal sheaf on

$$\begin{tikzcd} & X \times I_n(X,\beta) \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \\ X & I_n(X,\beta) \end{tikzcd}$$

$\gamma \in H^{\bullet}(X, \mathbb{Z})$, $k \geq 0$,

$$ch_{2+k}(\gamma) : H_{\bullet}(I_n(X,\beta), \mathbb{Q}) \longrightarrow H_{\bullet}(I_n(X,\beta), \mathbb{Q})$$

$$ch_{2+k}(\gamma)(-) = \pi_{2*}\left(ch_{2+k}(\mathbb{I})\, \pi_1^*(\gamma) \cap \pi_2^*(-)\right).$$

Given $\gamma_1, \ldots, \gamma_r \in H^{\bullet}(X)$, $k_1, \ldots, k_r \geq 0$,

$$\langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_r}(\gamma_r) \rangle_{\beta,n} := \int_{[I_n(X,\beta)]^{vir}} ch_{2+k_1}(\gamma_1) \cdots ch_{2+k_r}(\gamma_r) \in \mathbb{Z}.$$

$$DT_\beta^X(q; \underline{\gamma}, \underline{k}) := \sum_{n \in \mathbb{Z}} \langle \tau_{k_1}(\gamma_1), \ldots, \tau_{k_r}(\gamma_r) \rangle_{\beta,n}\, q^n$$

$$DT^X(q; \gamma, k) := \sum_{\beta} DT_\beta^X(q; \gamma, k)\, q^\beta.$$

Issue:

Issue. DT thy of pts contributes to DT thy class β.

β can be zero, $I_n(X,\beta)$: Hilbert sch of pts on X.

edim $= \beta \cdot c_1(X) = 0$.

no insertions.

$[I_n(X,0)]^{vir} \in \mathbb{Z}$.

Better to study $\boxed{\dfrac{DT^X_\beta(q;\gamma,k)}{DT^X_0(q)}}$.

$$DT^X_0(q) = \left(\prod_{n\geq 1} (1-q^n)^{-n}\right)^{\chi(X)}$$

MacMahon's fct

Conj (Maulik-Nekrasov-Okounkov-Pandharipande)

GW thy $\longleftrightarrow$ DT thy

$$\frac{DT^X_\beta(q;y,k)}{DT^X_0(q)}$$

rat'l fct in q.

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2) stable pairs: Pandharipande-Thomas thy

$$\mathcal{O}_X \xrightarrow{s} F$$

F pure 1 dim'l shv

$[F] = \beta \in H_2(X, \mathbb{Z})$,

$\chi(F) = n$

coker(s) zero dim'l.

$P_n(X,\beta)$ moduli of stable pairs

$[P_n(X,\beta)]^{vir} \in A_{edim}(P_n(X,\beta))$

$PT^X_\beta(q;y,k)$, $PT^X(q;y,k)$.

$$0 \to \mathcal{O}_D \longrightarrow F \longrightarrow Q \longrightarrow 0$$

$\mathcal{O}_D$ = im(s)

D Cohen-Mac curve

Q = coker(s)

zero dim'l on D

D

Conj (Pandharipande - Thomas)

$$PT^X_\beta(q;y,k) = \frac{DT^X_\beta(q;y,k)}{DT^X_0(q)}.$$

Known for X Calabi-Yau 3fold : $\omega_X \cong \mathcal{O}_X$.
not known beyond that.
proof for CY3: Bridgeland, Toda using Joyce's motivic Hall algebra.
(wall crossing thy of Kontsevich-Soibelman)

Behrend fct $\nu : I_n(X,\beta) \longrightarrow \mathbb{Z}$ $\quad$ $C \in CY3$

$$[I_n(X,\beta)]^{vir} = \sum_{a\in\mathbb{Z}} a\,\chi(\nu^{-1}(a))$$

$I_n(X,\beta) \ni P$ locally near P, $I_n(X,\beta)$ is crit(f : smooth $\to \mathbb{C}$)

$\nu(P) = $ sign $\chi(\varphi_{f,P})$

3) relative stable pairs

$f : Y \to X$ $\quad Y, X$ projective threefolds

Y smooth

$Rf_* \mathcal{O}_Y = \mathcal{O}_X$.

Bryan-Steinberg : $\mathcal{X}$ CY3 orbifolds (e.g. $\mathbb{C}^3/\mathbb{Z}/n \ (k,-k,0)$)

$$\begin{tikzcd} & \mathcal{X} \arrow[d] & \\ Y \arrow[r, "\text{crepant resolution}"] & X & \text{coarse space} \end{tikzcd}$$

$D^b(\mathcal{X}) \cong D^b(Y)$

crepant resolution conj:

$$PT^{\mathcal{X}}(q) = \frac{DT^{\mathcal{X}}(q)}{DT_0^{\mathcal{X}}(q)} = \boxed{\frac{DT^{Y}(q)}{DT^{Y}_{exc}(q)}} = BS^{f}(q).$$

$$DT^{Y}_{exc}(q) = \sum_{\beta \text{ exc class}} DT^{Y}_{\beta}(q)$$

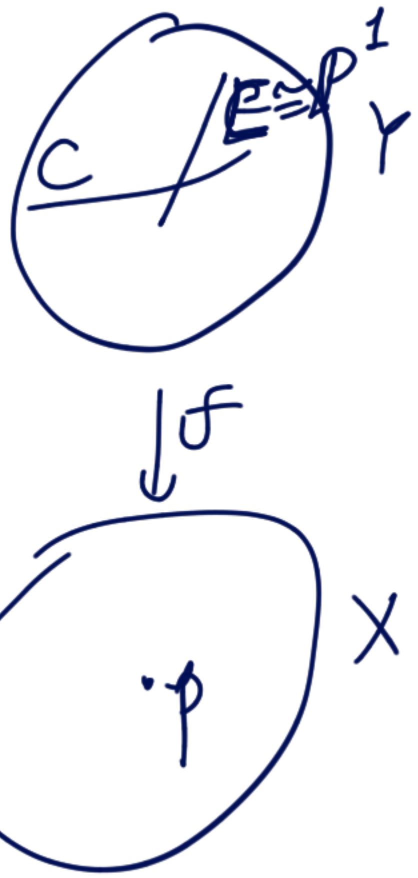
$\mathcal{O}_Y \xrightarrow{s} F$ $\qquad\qquad$ $F \in \mathbb{F}$

$\mathrm{coker}(s) \in \mathbb{T}$

$\mathbb{T}$: Q shvs on Y s.t. $Rf_*Q = 0$ or it is a zero dim'l sheaf on X

$Q \in Coh_{\leq 1}(Y)$ $\qquad$ $R^if_*Q = 0, i > 0$.

$\mathbb{F}$: R shvs in $Coh_{\leq 1}(Y)$ with $Hom(Q, R) = 0$ for $Q \in \mathbb{F}$.



$\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_{C \cup E}$

DT, PT on Y

but it is not a BS pair on Y

$$0 \to \underbrace{\boxed{\mathcal{O}_E(-1)}}_{\in \mathbb{T}} \to \mathcal{O}_{C \cup E} \to \mathcal{O}_C \to 0$$

But F non-triv ext

$$\begin{tikzcd} & \mathcal{O}_Y \arrow[d] \arrow[dr, "s"] & & & \\ 0 \arrow[r] & \mathcal{O}_C \arrow[r] & F \arrow[r] & \underline{\mathcal{O}_E(-1)} \arrow[r] & 0 \end{tikzcd}$$

BS pair

Bryan–Steinberg: defined invs using Behrend fct for Y CY_3

$$BS^f(q) = \frac{DT^Y(q)}{DT^Y_{exc}(q)} = \frac{PT^Y(q)}{PT^Y_{exc}(q)}.$$

Thm (P). There are proper alg spaces $BS^f_n(Y,\beta)$ with natural virtual fundamental classes

$$[BS^f_n(Y,\beta)]^{vir} \in A_{edim}(BS^f_n(Y,\beta)).$$

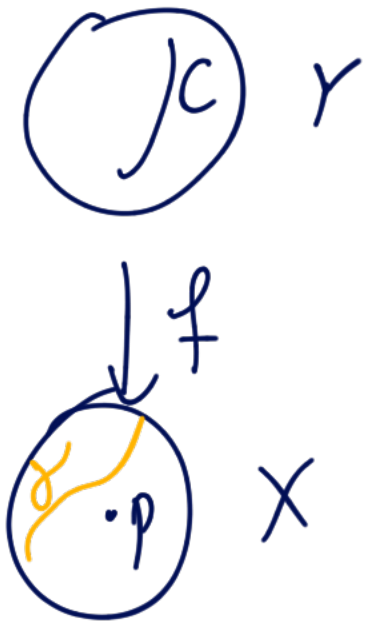
$BS^f_\beta(q;\gamma,k)$, $BS^f(q;\gamma,k)$.

Conj. $$BS^f(q;\gamma,k) = \frac{PT^Y(q;\gamma,k)}{PT^Y_{exc}(q)} \qquad \frac{\cdots}{DT^X_0(q)}$$

when $\gamma \in f^*H^{\geq 4}(X)$.

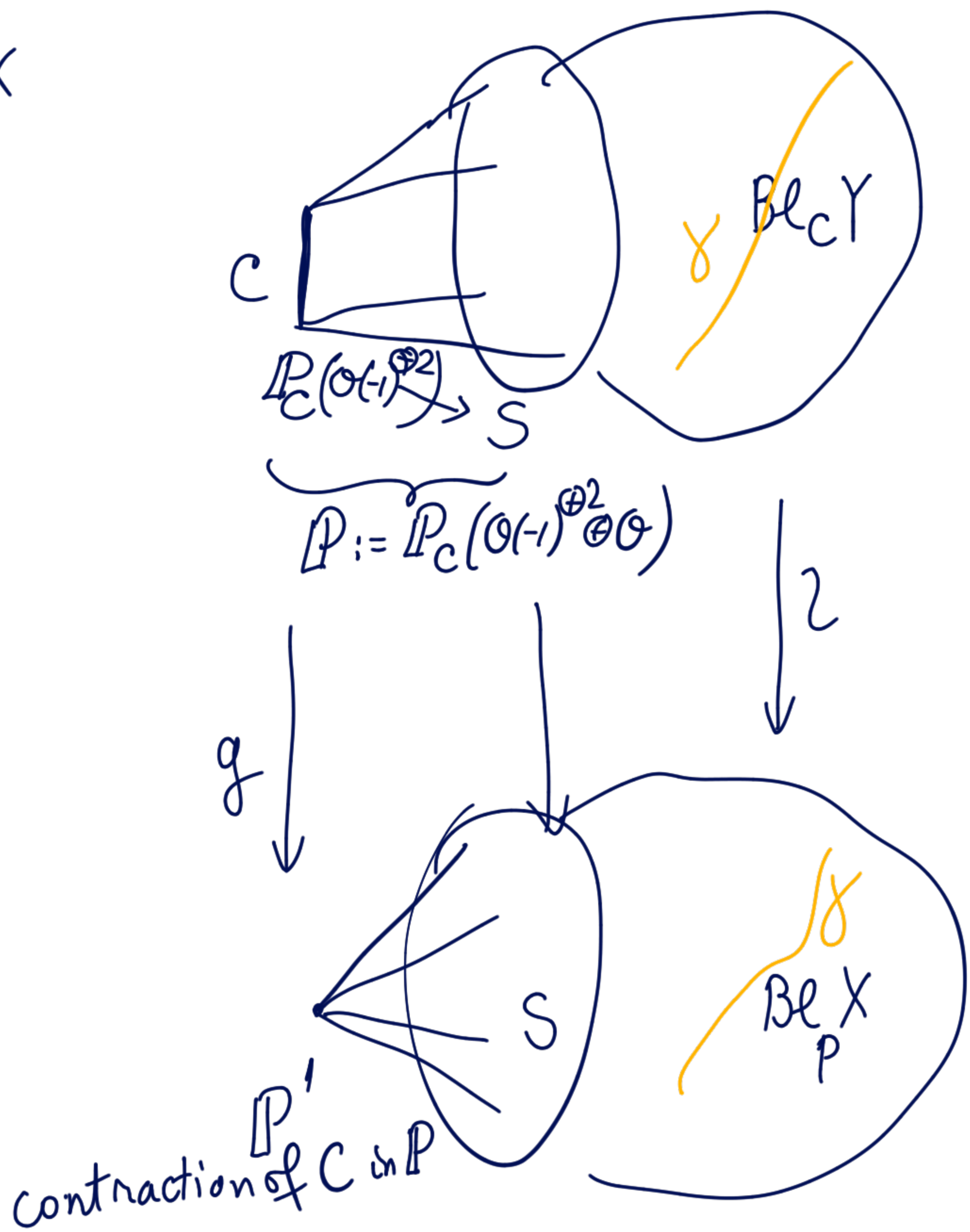
Thm (P). Conj is true for $f: Y \to X$ contraction of $C \cong \mathbb{P}^1$ with $N_{C/Y} \cong \mathcal{O}(-1)^{\oplus 2}$.

1) Degeneration formula



$$\begin{tikzcd} Bl_{C\times 0}(Y\times\mathbb{C}) \arrow[rr] \arrow[dr] & & Bl_{p\times 0}(X\times\mathbb{C}) \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

over 0 :



$$PT^{Y}_{\beta}(q;y,k) = \sum \boxed{PT^{\mathbb{P}/S}_{\beta_1,\eta}(q)}\, \boxed{PT^{Bl_C Y/S}_{\beta_2,\eta}(q;y,k)}\, \mathrm{fact}(\eta)$$

$$BS^{f}_{\beta}(q;y,k) = \sum \boxed{BS^{g/S}_{\beta_1,\eta}(q)}\, \boxed{PT^{Bl_C Y/S}_{\beta_2,\eta}(q;y,k)}\, \mathrm{fact}(\eta)$$

η

bubbles $\mathbb{P}_S(N_S \oplus O)$

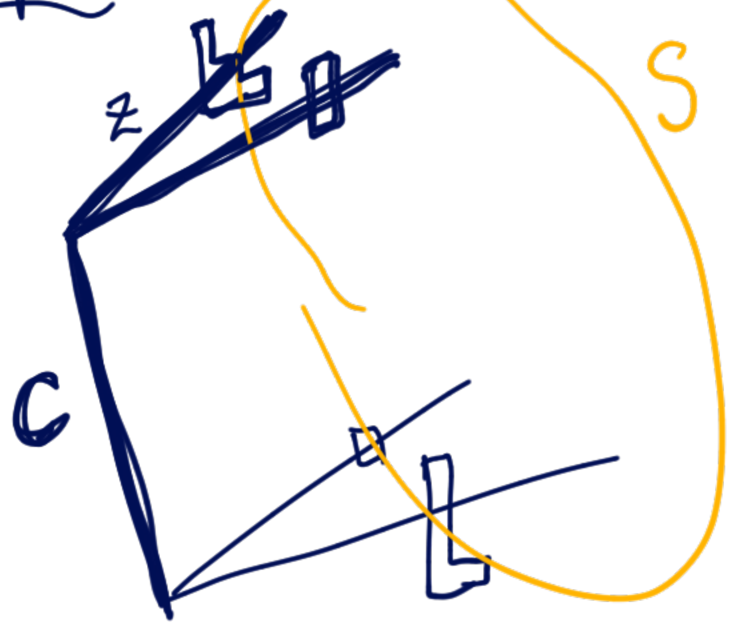
$$\varepsilon : BS^{g/S}_{n}(\mathbb{P},\beta) \longrightarrow Hilb(S,k)$$

F $\qquad \beta . S = k.$

$\eta \rightsquigarrow$ Cohomology classes on $Hilb(S,k)$.

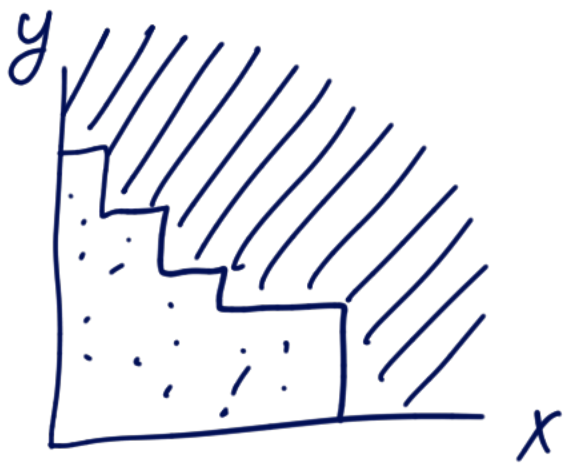
$$BS^{g/S}_{\beta_1,\eta}(q) = \sum_{n} \left(\int_{[BS^{g/S}_{n}(\mathbb{P},\beta_1)]^{vir}} \varepsilon^*(\eta) \right) q^n$$

Step 2. localization theorem for $T \simeq (\mathbb{C}^*)^2$



preserves nat $\mathcal{O}$ form on $P \setminus S = \mathrm{Tot}_C\, \mathcal{O}(-1)^{\oplus 2}$

$BS, PT \,\big|_{P \setminus C}$



$$\pi = (\pi_i)_{i=1}^{4}$$

$$\beta = [\pi] + m[C] \in H_2(P, \mathbb{Z})$$

$BS_n(\pi, m)$ $\qquad PT_n(\pi, m)$.

$$\int_{[BS_n(\pi,m)]^{vir}} \frac{\mathcal{E}^*(\eta)}{e(N^{vir})} \qquad \int_{[PT_n(\pi,m)]^{vir}} \frac{\mathcal{E}^*(\eta)}{e(N^{vir})}$$

It is enough to compare $[BS_n(\pi,m)]^{vir}$ & $[PT_n(\pi,m)]^{vir}$.

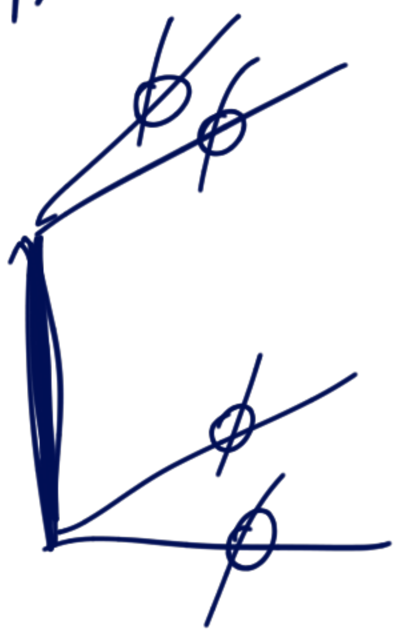
Step 3. These classes have edim zero.

can use motivic Hall algs. to finish the conjecture.

$BS_\pi(q) \qquad PT_\pi(q)$

$$BS_\pi(q) = \frac{PT_\pi(q)}{PT_0(q)}.$$

$\uparrow$



$\mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z)$

$\mathrm{Ext}^3(\mathcal{O}_Z, \mathcal{O}_Z)$