**AG Venjakob**

Hauptseminar $p$-adische Arithmetische Geometrie
WS 21/22

**Bloch-Kato Selmer groups and the Fargues-Fontaine curve**

**Time:** Thursday 11:15 **Place:** SR 8, hybrid, i.e. with possibilities to join online

In the Iwasawa theory of motives (see e.g. [FK]) Selmer groups attached to the $\lambda$-adic realisations of a given motive $M$ are linked to (conjecturally existing) $p$-adic $L$-functions attached to $M$. The defining local conditions of such global Selmer groups are usually given by means of $p$-adic Hodge theory, viz by Bloch's and Kato's local Galois cohomology groups which we shall shortly recall now: Let $K$ be a finite extension of $\mathbb{Q}_p$ and $V$ a $p$-adic representation of the absolute Galois group $G_K$ of $K$ with Galois stable $\mathbb{Z}_p$-lattice $T$. For any algebraic extension $L/K$ Bloch and Kato [BK] define the following subgroups

$$H^1_e(L, V/T) \subseteq H^1_f(L, V/T) \subseteq H^1_g(L, V/T) \subseteq H^1(L, V/T)$$

of the first continuous group cohomology $H^1(L, V/T) = H^1(G_L, V/T)$, which are involved in their conjectures on special values of $L$-functions of motives. E.g. if $T = T_p(A)$ is the $p$-adic Tate module of an Abelian variety $A$ defined over $K$, then all these distinguished subgroups coincide and are equal to the image of the Kummer map

$$\kappa_L : A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1(L, A(p)),$$

where now the $p$-primary torsion points $A(p)$ are naturally identified with $V/T$.

In Iwasawa theory, a precise description of the Bloch-Kato subgroups when $L$ is an infinite extension of $K$ is essential to study the Selmer groups of motives over infinite extensions of number fields (notably to prove so-called "control theorems") and it can also be used to construct $p$-adic height pairings. For Abelian varieties, the most general result is due to Coates and Greenberg [CG], who gave the following simple cohomological description: Let $V_0$ be the minimal sub-$G_K$-representation of $V$ such that the Hodge-Tate weights of $V/V_0$ are all less than or equal to 0, and $T_0 := T \cap V_0$. Then the inclusion induces a map

$$\lambda_L : H^1(L, V_0/T_0) \to H^1(L, V/T)$$

and Coates and Greenberg prove that

$$\mathrm{Im}(\kappa_L) = \mathrm{Im}(\lambda_L)$$

for $L$ being *deeply ramified*. Nowadays one would rephrase the latter condition by requiring that $\hat{L}$ is a *perfectoid field* in the sense of Scholze. The aim of this seminar is to understand Ponsinet's generalisation [P] of this nice result to more general de Rham representations $V$. If the HT-weights of $V$ are less than or equal to 1, and $\hat{L}$ is a *perfectoid field*, then

$$H^1_e(L, V/T) = \mathrm{Im}(\lambda_L).$$

Furthermore, without the restriction of the HT-weights he establishes an explicit description of $\mathrm{Im}(\lambda_L)/H^1_e(L, V/T)$ in terms of objects in $p$-adic Hodge theory related to $B$-pairs à la Berger, which in turn are related to vector bundles over the Fargues-Fontaine curve. Thus the strategy for the proof relies on a close inspection of the Harder-Narasimhan filtration for such vector bundles.

**Talks (each topic for 2 x 90 minutes)**

1. (Bloch-Kato subgroups and Selmer groups) Recall period rings from 1.1 (as black box) with fundamental exact sequence together with the definition of $H^1_?(L, -)$ of [BK] (see also [P] §3.2), explain properties for $p$-divisible groups and Abelian varieties and behaviour under local Tate duality. Discuss the difference between BK and Greenberg Selmer groups (or complexes) from [FK]. See also §4.2 of [V].
2. (The results of Coates-Greenberg) Report on [CG]
3. (The Fargues-Fontaine curve and vector bundles) [P] §1.2-7
4. (Truncation of the Hodge-Tate filtration) [P] §2
5. (Groups of points) §3.1-3
6. (Cohomology of perfectoid fiels and univeral norms) §3.4-5

# References


[CG] Coates, J.; Greenberg, R.: *Kummer theory for abelian varieties over local fields.* Invent. Math. 124 (1996), no. 1-3, 129–174.

[P] Gautier Ponsinet.: *Universal norms and the Fargues-Fontaine curve*. Preprint 2020. arXiv:2010.02292 `https://arxiv.org/abs/2010.02292`

[FK] Fukaya, Takako; Kato, Kazuya.: *A formulation of conjectures on $p$-adic zeta functions in noncommutative Iwasawa theory.* Proceedings of the St. Petersburg Mathematical Society. Vol. XII, 1–85, Amer. Math. Soc. Transl. Ser. 2, 219, Amer. Math. Soc., Providence, RI, 2006.

[BK] Bloch, Spencer; Kato, Kazuya.: *$L$-functions and Tamagawa numbers of motives.* The Grothendieck Festschrift, Vol. I, 333–400, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[V] Venjakob, Otmar.: *On the Iwasawa theory of $p$-adic Lie extensions.* Compositio Math. 138 (2003), no. 1, 1–54.