On the Surface Displacement of an Isotropic Elastic Halfspace Containing an Inextensible Membrane Reinforcement

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Dedicated to Professor David M. Barnett

Abstract: This paper examines the axisymmetric problem of the uniform circular surface loading of an isotropic elastic halfspace, which is internally reinforced with an inextensible membrane of finite extent.

Key Words: reinforced halfspace, elastic deformations, inextensible reinforcement, mixed boundary value problem, dual integral equations, Fredholm integral equations, surface displacement

1. INTRODUCTION

The mathematical modeling of the problem of an elastic halfspace region that is reinforced by a closely spaced distribution of inextensible reinforcing layers has a long history dating back to a classical study by H. M. Westergaard [1]. Analogous expositions of constrained media are also given by Wieghardt [2] and Vlazov [3] (see, for example, Selvadurai [4]). Modern continuum treatments of elastic media with inextensibility constraints provided by a distribution of reinforcing fibres are also given by Spencer [5]. With the current use of geosynthetic reinforcing materials, there is continuing interest in the modeling of such problems with reference to the evaluation of both the deformability characteristics and load carrying capacity of reinforced geomaterials. In this paper we consider the *axisymmetric* problem of the surface loading of an isotropic elastic halfspace, which is reinforced with an inextensible membrane located at a finite depth below its surface. In the case where the membrane exhibits deformability characteristics consistent with extensibility and flexure, the resulting boundary value problem can be approached using the methodologies presented here, and such a problem will be a useful model of an embedded sensor located in an elastic solid described by Barnett et al. [6]. The *a priori* imposition of an inextensibility constraint in the entire plane of the reinforcing membrane is recognized as a limitation in the modeling; nonetheless, it provides, as does the use of the classical theory of elasticity (Davis and Selvadurai [7], Selvadurai [8]), a useful first approximation for the study of this class of problems. In reality, how-

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ever, the inextensibility constraint will be meaningful only in regions where the membrane exhibits tensile in-plane forces. The specific problem examined in this paper deals with the axisymmetric normal loading of a membrane-reinforced halfspace by a uniform circular load of finite radius b. The general axisymmetric boundary value problem associated with the surface loading of the membrane-reinforced halfspace is developed by first considering a reinforcing region of finite radius a. This problem in elasticity can be effectively reduced to the solution of a Fredholm integral equation of the second kind. In this study, however, attention is restricted to the specific situation where the inextensible reinforcing region is of *infinite extent*. In this case the influence of the reinforcing action by the inextensible membrane can also be assessed through a set of limiting estimates.

2. GOVERNING EQUATIONS

The axisymmetric problem in isotropic elasticity can be examined by making use of a number of approaches, including the strain potential approach proposed by Love and the displacement function approaches proposed by Boussinesq, Neuber, Papkovich and others (see, e.g. Truesdell [9], Gurtin [10], Gladwell [11], and Barber [12]). Following Green & Zerna [13], the solution to the axisymmetric problem in classical elasticity, referred to the cylindrical polar coordinate system, can be expressed in terms of two harmonic functions $\varphi\,(r, z)$ and $\psi\,(r, z)$ which satisfy

$$\nabla^2\varphi(r, z) = 0; \quad \nabla^2\psi(r, z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The displacement and stress components referred to the cylindrical polar coordinate system can be expressed in terms of $\varphi\,(r, z)$ and $\psi\,(r, z)$ as follows:

$$u_r = \frac{\partial\varphi}{\partial r} - z\frac{\partial\psi}{\partial r} \tag{3}$$

$$u_z = (3-4\nu)\psi + \frac{\partial\varphi}{\partial z} - z\frac{\partial\psi}{\partial z} \tag{4}$$

and

$$\sigma_{rr} = 2\mu\left[\frac{\partial^2\varphi}{\partial r^2} + 2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial r^2}\right] \tag{5}$$

$$\sigma_{\theta\theta} = 2\mu\left[\frac{1}{r}\frac{\partial\varphi}{\partial r} + 2\nu\frac{\partial\psi}{\partial z} - \frac{z}{r}\frac{\partial\psi}{\partial r}\right] \tag{6}$$

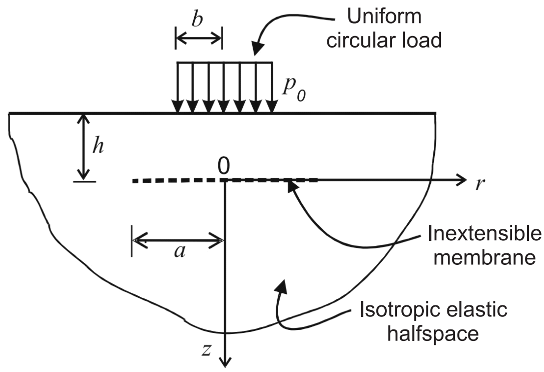

Figure 1. The axisymmetric loading of a halfspace containing an inextensible circular membrane.

$$\sigma_{zz} = 2\mu\left[2(1-\nu)\frac{\partial\psi}{\partial z} + \frac{\partial^2\varphi}{\partial z^2} - z\frac{\partial^2\psi}{\partial z^2}\right] \tag{7}$$

$$\sigma_{rz} = 2\mu\left[(1-2\nu)\frac{\partial\psi}{\partial r} + \frac{\partial^2\varphi}{\partial r\partial z} - z\frac{\partial^2\psi}{\partial r\partial z}\right] \tag{8}$$

where μ is the linear elastic shear modulus and ν is Poisson's ratio. Owing to axial symmetry, all other displacement and stress components are zero.

3. THE BOUNDARY VALUE PROBLEM

Consider the problem of an isotropic elastic halfspace region, which contains a circular inextensible membrane of radius a, and is located at a finite depth h from the surface of the halfspace. The surface of the halfspace region ($z = -h$) is subjected to a uniform compressive normal circular load of radius b and stress intensity p_0 such that the resulting state of deformation is axisymmetric (Figure 1). We further assume that the membrane region exerts an inextensibility constraint over the entire region $r \in (0, a)$. The origin of coordinates is located at the center of the inextensible membrane. The mixed boundary value problem associated with the membrane reinforced halfspace problem is as follows:

$$\sigma_{zz}^{(1)}(r,-h) = \begin{cases} -p_0; & 0 < r < b \\ 0; & b < r < \infty \end{cases} \tag{9}$$

$$\sigma_{rz}^{(1)}(r,-h) = 0; \quad 0 \le r < \infty \tag{10}$$

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0) = 0; \quad 0 \le r \le a \tag{11}$$

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0); \quad a \le r < \infty \tag{12}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0); \quad 0 \le r < \infty \tag{13}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0); \quad 0 < r < \infty \tag{14}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0); \quad a < r < \infty \tag{15}$$

where, the superscripts $^{(1)}$ and $^{(2)}$ refer, respectively, to the layer region ($r \in (0,\infty)$; $z \in (0,-h)$) and the halfspace region ($r \in (0,\infty)$; $z \in (0,\infty)$).

By considering a Hankel transform development of the governing partial differential equations (1) we can obtain solutions separately applicable to the layer and halfspace regions. These can be used in equations (2) to (8) to generate the necessary expressions for the displacement and stress components. Considering the layer region $^{(1)}$ we obtain Hankel integral solutions (see e.g. Sneddon [14]), that are indeterminate to within four arbitrary functions of an integration parameter. We can eliminate two of these functions by satisfying the traction boundary conditions on the surface $z = -h$. The resulting expressions for the displacements and stresses in the elastic layer can be written as follows:

$$\begin{aligned} u_r^{(1)}(r,z) &= \int_0^\infty \Big\{ [C(\xi)\,\xi\,(z+h) - 2(1-\nu)\,D(\xi)]\cosh(\xi[z+h]) \\ &+ [(1-2\nu)\,C(\xi) - \xi\,(z+h)\,D(\xi)]\sinh(\xi[z+h]) \\ &- X(\nu,\xi h)\,\frac{\cosh(\xi[z+h])}{(1-2\nu)}\Big\}\frac{J_1(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \end{aligned} \tag{16}$$

$$\begin{aligned} u_z^{(1)}(r,z) &= \int_0^\infty \Big\{ [2(1-\nu)\,C(\xi) + \xi\,(z+h)\,D(\xi)]\cosh(\xi[z+h]) \\ &- [\xi\,(z+h)\,C(\xi) + (1-2\nu)\,D(\xi)]\sinh(\xi[z+h]) \\ &+ X(\nu,\xi h)\,\frac{\sinh(\xi[z+h])}{(1-2\nu)}\Big\}\frac{J_0(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \end{aligned} \tag{17}$$

$$\begin{aligned} \sigma_{zz}^{(1)}(r,z) &= 2\mu\int_0^\infty \Big\{ [C(\xi) + \xi\,(z+h)\,D(\xi)]\sinh(\xi[z+h]) \\ &- \xi\,(z+h)\,C(\xi)\cosh(\xi[z+h]) \\ &+ X(\nu,\xi h)\,\frac{\cosh(\xi[z+h])}{(1-2\nu)}\Big\}\frac{\xi J_0(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \end{aligned} \tag{18}$$

$$\begin{aligned} \sigma_{rz}^{(1)}(r,z) &= -2\mu\int_0^\infty \Big\{ [D(\xi) - \xi\,(z+h)\,C(\xi)]\sinh(\xi[z+h]) \\ &+ \xi\,(\xi+h)\,D(\xi)\cosh(\xi[z+h]) \\ &+ X(\nu,\xi h)\,\frac{\sinh(\xi[z+h])}{(1-2\nu)}\Big\}\frac{\xi J_1(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \end{aligned} \tag{19}$$

where $C(\xi)$ and $D(\xi)$ are arbitrary functions and

$$X(\nu, \xi h) = -\frac{p_0 b (1-2\nu)}{2\mu} \left[\frac{\sinh(\xi h) J_1(\xi b)}{\xi} \right]. \tag{20}$$

For the halfspace region, we can select the Hankel integral solutions such that the displacement and stress fields satisfy the regularity conditions pertaining to their decay to zero as $(r, z) \to \infty$. The relevant Hankel integral solutions for the displacements and stresses take the forms

$$\begin{aligned} u_r^{(2)}(r,z) &= \int_0^\infty [A(\xi) + \xi z B(\xi)] \mathrm{e}^{-\xi z} J_1(\xi r)\, \mathrm{d}\xi \\ u_z^{(2)}(r,z) &= \int_0^\infty [A(\xi) + (3-4\nu) B(\xi) + \xi z B(\xi)] \mathrm{e}^{-\xi z} J_0(\xi r)\, \mathrm{d}\xi \end{aligned} \tag{21}$$

and

$$\begin{aligned} \sigma_{zz}^{(2)}(r,z) &= -2\mu \int_0^\infty \xi [A(\xi) + 2(1-\nu) B(\xi) + \xi z B(\xi)] \mathrm{e}^{-\xi z} J_0(\xi r)\, \mathrm{d}\xi \\ \sigma_{rz}^{(2)}(r,z) &= -2\mu \int_0^\infty \xi [A(\xi) + (1-2\nu) B(\xi) + \xi z B(\xi)] \mathrm{e}^{-\xi z} J_1(\xi r)\, \mathrm{d}\xi \end{aligned} \tag{22}$$

Considering the inextensibility constraint (11) and the continuity conditions (12), (13) and (14), we have

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0); \quad 0 < r < \infty \tag{23}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0); \quad 0 < r < \infty \tag{24}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0); \quad 0 < r < \infty. \tag{25}$$

Using the relations Equations (16)–(19), (21) and (22), we can obtain expressions for $A(\xi)$, $B(\xi)$ and $D(\xi)$ in terms of $C(\xi)$ and $X(\nu, \xi h)$. Avoiding the details of calculations, it can be shown that the resulting expressions for $u_r^{(1)}(r,0)$, and the difference between the shear stress components at the plane of the membrane $\{\sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0)\}$, take the following forms:

$$\begin{aligned} u_r^{(1)}(r,0) &= \int_0^\infty \Bigg\{ C(\xi) [\xi h \coth(\xi h) + 1 - 2\nu] \\ &- \alpha_1(\nu, \xi h) [2(1-\nu) \coth(\xi h) + \xi h] \\ &- X(\nu, \xi h) \left[\frac{\coth(\xi h)}{(1-2\nu)} + \gamma_1(\xi) [\xi h + 2(1-\nu) \coth(\xi h)] \right] \Bigg\} J_1(\xi r)\, \mathrm{d}\xi \end{aligned} \tag{26}$$

and

$$\frac{(3-4\nu)\left\{\sigma_{rz}^{(1)}(r,0)-\sigma_{rz}^{(2)}(r,0)\right\}}{2\mu}$$

$$= -\Bigg[\int_0^\infty \xi C(\xi)\Bigg\{(3-4\nu)\Bigg[\alpha_1(\nu,\xi h)\ \{1+\xi h\coth(\xi h)\}$$

$$- \xi h - \{\xi h\coth(\xi h)+1-2\nu\}\,\alpha_1(\nu,\xi h)\,\{\xi h+2(1-\nu)\coth(\xi h)\}\Bigg]$$

$$- (1-2\nu)\Bigg[2(1-\nu)\coth(\xi h)-\xi h\{1+\coth(\xi h)\}-(1-2\nu)$$

$$+ \alpha_1(\nu,\xi h)\Bigg[\xi h[1+\coth(\xi h)]-(1-2\nu)+2(1-\nu)\coth(\xi h)\Bigg]\Bigg]\Bigg\}J_1(\xi r)\,\mathrm{d}\xi$$

$$+ \frac{2\mu}{(3-4\nu)}\int_0^\infty \xi X(\nu,\xi h)\Bigg\{(3-4\nu)\Bigg[\frac{-1}{(1-2\nu)}[1+\coth(\xi h)]$$

$$- \gamma_1(\nu,\xi h)[1+\xi h\coth(\xi h)]+2(1-\nu)\coth(\xi h)\Bigg]$$

$$+ (1-2\nu)\gamma_1(\nu,\xi h)\Bigg[\xi h[1+\coth(\xi h)]-(1-2\nu)+2(1-\nu)\coth(\xi h)$$

$$+ \frac{[1+\coth(\xi h)]}{(1-2\nu)\gamma_1(\nu,\xi h)}\Bigg]J_1(\xi r)\,\mathrm{d}\xi \tag{27}$$

where

$$\alpha_1(\nu,\xi h)=\frac{2-2\nu-\xi h}{1-2\nu-\xi h};\qquad \gamma_1(\nu,\xi h)=\frac{-1}{(1-2\nu)(1-2\nu-\xi h)}. \tag{28}$$

The results (26) and (27) can now be used in the interface conditions (11) and (15) to generate a pair of dual integral equations for the unknown function $C(\xi)$. Through the introduction of substitution functions, these dual integral equations can be reduced to the forms

$$\int_0^\infty F(\nu,\xi h)\,C^*(\xi)\,J_1(\xi r)\,\mathrm{d}\xi = L(\nu,\xi h);\quad 0<r<a \tag{29}$$

$$\int_0^\infty \xi C^*(\xi)\,J_1(\xi r)\,\mathrm{d}\xi = 0;\quad a<r<\infty \tag{30}$$

where $C^*(\xi)$ is directly related to $C(\xi)$, and $F(\nu,\xi h)$ and $L(\nu,\xi h)$ are known functions. The method of solution of the system of dual integral equations defined by (29) and (30) is now relatively well known (see, for example, Sneddon [14, 15], Gladwell [11]). The method involves the introduction of a finite Fourier transform such that the integral equation (30) is identically satisfied, i.e.

$$C^*(\xi) = \int_0^a \Phi(t)\sin(\xi t)\mathrm{d}t \tag{31}$$

where $\Phi(t)$ is an arbitrary function. The integral equation (29) can now be reduced to a *Fredholm integral equation of the second kind* for the unknown function $\Phi(t)$, i.e.

$$\begin{aligned}
& t\Phi(t) - 4(1-\nu)\int_0^a \Phi(u)\,K(u,t)\,\mathrm{d}u \\
= \; & \frac{8(1-\nu)t}{\pi}\int_0^\infty X(\nu,\xi h)\left\{-\frac{\coth(\xi h)}{(1-2\nu)} - \gamma_1(\nu,\xi h)\,[\xi h + 2(1-\nu)\coth(\xi h)]\right. \\
+ \; & \left. a_2(\nu,\xi h)\left[M(\nu,\xi h) - \frac{1}{4(1-\nu)}\right]\right\}\sin(\xi t)\,\mathrm{d}\xi, \quad 0 < t < a
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
M(\nu,\xi h) &= \frac{1}{4(1-\nu)} + \frac{[\{1-2\nu+\xi h\coth(\xi h)\} - \alpha_1(\xi)\{\xi h + 2(1-\nu)\coth(\xi h)\}]}{N(\nu,\xi h)} \\
N(\nu,\xi h) &= (3-4\nu)\Big\{\alpha_1(\nu,\xi h)\,[1+\xi h\coth(\xi h)+\xi h+2(1-\nu)\coth(\xi h)] \\
&\quad - [1-2\nu+\xi h+\coth(\xi h)]\Big\} \\
&\quad - (1-2\nu)\Big\{\alpha_1(\nu,\xi h)\,[\xi h+\xi h\coth(\xi h)-(1-2\nu)+2(1-\nu)\coth(\xi h)] \\
&\quad + [2(1-\nu)\coth(\xi h)-(1-2\nu)-\xi h-\xi h\coth(\xi h)]\Big\}
\end{aligned} \tag{33}$$

$$\begin{aligned}
\alpha_2(\nu,\xi h) &= (3-4\nu)\Big\{-\frac{[1+\coth(\xi h)]}{(1-2\nu)} \\
&\quad - \gamma_1(\nu,\xi h)\,[1+\xi h\{1+\coth(\xi h)\}+2(1-\nu)\coth(\xi h)]\Big\} \\
&\quad + (1-2\nu)\,\gamma_1(\nu,\xi h)\Big\{\xi h\,[1+\coth(\xi h)]-(1-2\nu)+2(1-\nu)\coth(\xi h) \\
&\quad + \frac{[1+\coth(\xi h)]}{(1-2\nu)\,\gamma_1(\nu,\xi h)}\Big\}
\end{aligned} \tag{34}$$

and the kernel function $K\,(u,t)$ is given by

$$K(u,t) = \frac{2t}{\pi}\int_0^\infty M\,(\nu,\xi h)\sin(\xi u)\sin(\xi t)\,\mathrm{d}\xi. \tag{35}$$

With the understanding that the Fredholm integral equation of the second kind, (32), can be solved, the analysis of the problem of the surface loading of an isotropic elastic halfspace reinforced with an embedded inextensible membrane is now formally complete. There are a number of methodologies for the solution of the Fredholm integral equation of the second-kind and these are documented by Baker [16], Delves and Mohamed [17] and Atkinson [18] and examples of the application of techniques that reduce the solution of the Fredholm Integral equation of the second kind to a matrix equation, where the unknown function is expressed as a vector of a discrete number of unknowns, are given by Selvadurai [19–21] and Selvadurai et al. [22]. The results for the displacements and stresses in the halfspace region can be obtained by appropriate back-substitutions.

4. NUMERICAL ESTIMATES

An inspection of the Fredholm integral equation of the second-kind (32) clearly indicates that owing to the form of functions $M(\nu,\xi h)$, $N(\nu,\xi h)$, etc., this equation is not amenable to *exact solution*. Recourse must therefore be made to its solution in a numerical fashion. The numerical procedure for the solution of the governing integral equation is discussed in detail in the references cited previously. For the present we shall discuss some limiting cases of particular interest to engineering applications.

We focus attention on the specific problem of an isotropic elastic halfspace region, which is reinforced with an *inextensible flexible membrane of infinite extent,* located at a finite depth h below its surface. This corresponds to the limiting case of the present general formulation where $a \to \infty$. The surface of the halfspace is subjected to a uniform circular load of stress intensity p_0. The result of primary interest invariably relates to examining the influence of the embedded inextensible membrane of infinite extent on the surface displacement of the halfspace region. In this instance, the analysis is considerably simplified and the function $C(\xi)$ is given by the expression

$$C\,(\xi) = \frac{X\,(\xi)}{\Delta\,(\nu,\xi h)}\left[\frac{\coth(\xi h)}{(1-2\nu)} + \gamma_1(\xi)\,\{\xi h + 2\,(1-\nu)\coth(\xi h)\}\right] \tag{36}$$

where

$$\Delta\,(\nu,\xi h) = \{1-2\nu+\xi h\coth(\xi h)\} - \alpha_1(\nu,\xi h)\,\{2\,(1-\nu)\coth(\xi h)+\xi h\}\,. \tag{37}$$

The result of primary importance to the current discussion, namely the distribution of axial displacement at the surface of the halfspace, can be evaluated in the form

$$u_z^{(1)}(r,-h) = -2(1-\nu)\int_0^\infty C(\xi)\,\frac{J_0(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi. \tag{38}$$

In the particular instance when $h \to 0$, we obtain the special case where the inextensible membrane is located at the surface of the elastic halfspace and the result (38) reduces to

$$\left[u_z^{(1)}(r,-h)\right]_{h\to 0} = \frac{(3-4\nu)\,p_0 b}{4\mu\,(1-\nu)}\int_0^\infty \frac{1}{\xi} J_1(\xi b)\,J_0(\xi r)\,\mathrm{d}\xi. \tag{39}$$

This result is identical to the axisymmetric axial displacement at the plane $z = 0$ of an *infinite space* region that is subjected to a uniform circular load of stress intensity $2p_0$, which acts in the z-direction over an area of radius b (see also Timoshenko and Goodier [23], Westergaard [24], Gladwell [11], Barber [12], Selvadurai [25]). The similarity between Kelvin's solution for the concentrated force acting at the interior of an infinite space region and Boussinesq's solution for a concentrated normal force acting at the surface of a halfspace region now becomes self-evident. In the case of the Kelvin force problem the plane of asymmetry in essence serves as an inextensible membrane (Westergaard [24], Barber [12], Selvadurai [26]).

When $h \to \infty$, naturally the surface displacements of the halfspace region are uninfluenced by the inextensible membrane and will correspond to the result

$$\left[u_z^{(1)}(r,-h)\right]_{h\to\infty} = \frac{(1-\nu)\,pb}{\mu}\int_0^\infty \frac{1}{\xi} J_1(\xi b)\,J_0(\xi r)\,\mathrm{d}\xi. \tag{40}$$

The infinite integrals occurring in (38) and (39) can also be expressed in the form of *complete elliptic integrals* in the following form:

$$\begin{aligned}\int_0^\infty \frac{1}{\xi} J_1(\xi b) J_0(\xi r)\mathrm{d}\xi \;&=\; \left(\frac{2}{\pi}\right)\frac{r}{b}\left[-\int_0^{\frac{\pi}{2}}\left[1-\frac{r^2}{b^2}\sin^2\theta\right]^{\frac{1}{2}}\mathrm{d}\theta\right.\\ &\quad-\left.\left(1-\frac{r^2}{b^2}\right)\int_0^{\frac{\pi}{2}}\left[1-\frac{r^2}{b^2}\sin^2\theta\right]^{-\frac{1}{2}}\mathrm{d}\theta\right].\end{aligned} \tag{41}$$

In establishing these limiting cases we have effectively *bounded* the surface displacements of the halfspace region containing an inextensible membrane of infinite extent located at a finite depth below the surface of the halfspace region with arbitrary ν.

It is perhaps more important to note that in the particular case of an incompressible elastic material, $\nu = 1/2$, and both (39) and (40) converge to the same result, which indicates that in the case of *material incompressibility*, the provision of an inextensible flexible reinforcement in close proximity to the surface of the halfspace does not contribute in any way to mitigating the surface displacements of the elastic halfspace region. This of course does not preclude the existence of a specific interior positioning of the inextensible membrane, which would minimize the surface displacements. Such information can be obtained only by solving the general numerical solution of problem. Figure 2 illustrates the surface deflection profile for the general case involving an inextensible membrane of infinite extent located at

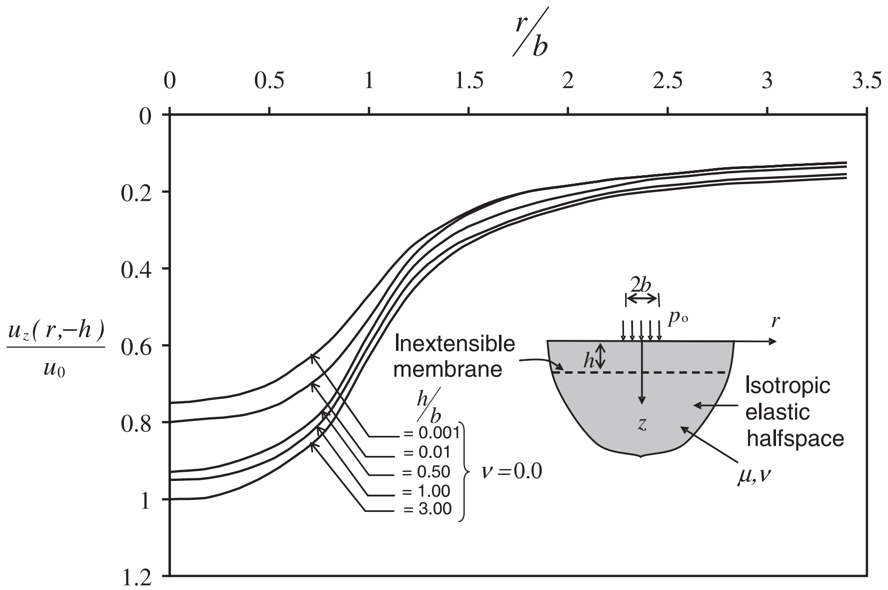

Figure 2. Surface displacements of the elastic halfspace containing an inextensible membrane of infinite extent [$u_0 = (1 - \nu)p_0 b/\mu$].

a finite depth below the surface of the halfspace region. The results are presented for the case where Poisson's ratio for the halfspace region $\nu = 0$, which represents the situation where the inextensible reinforcing layer would have the greatest influence in increasing the surface stiffness of the halfspace region. The study also indicates that the effectiveness of the inextensible membrane is realized only when its dimensions are substantially larger than the radius of the loaded region. Numerical solutions of the integral equations suggest that for the reinforcement to have any effect on the surface displacements, the radius of the reinforcing membrane must at least be in excess of three times the radius of the loaded area (i.e. $a > 3b$).

5. CONCLUDING REMARKS

The mechanical behavior of geomaterials that are internally reinforced by geosynthetic layers is complicated not only by the non-linear constitutive behavior of the geomaterial and the constitutive behavior of the membrane but also by the non-linear behavior of the interfaces that transmit the interaction between the geomaterials and the reinforcement. Simplified models only serve as guidelines to the possible limiting responses of the reinforced system. We have examined an idealized axisymmetric problem of the elastic behavior of a halfspace region that contains a circular membrane region, which is flexible but exhibits inextensibility. This constraint is assumed to apply irrespective of whether tensile or compressive net radial tractions act in the circular region containing the inextensible membrane. The greatest reduction in the surface settlements of the loaded halfspace region occurs when the inextensible membrane region is *bonded* to the surface of the elastic halfspace and for a Poisson's ratio $\nu = 0$. Also, for this value of Poisson's ratio, any effectiveness of the inextensible layer in reducing the surface settlements is realized only when $(h/b) < 3$, where h is the depth of embedment of the inextensible layer and b is the radius of the uniformly loaded circular region. Furthermore, when $\nu = 1/2$, the incorporation of a reinforcing membrane with inextensible behavior close to the surface of the halfspace results in no reduction of the surface displacements of the loaded halfspace region. Preliminary computations indicate that for material incompressibility, a reduction in the surface displacements is realized at an optimum depth of embedment $(h/b) \approx 1$.

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