International Journal of Engineering Science 47 (2009) 1339–1345

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Boussinesq indentation of an isotropic elastic halfspace reinforced with an inextensible membrane

A.P.S. Selvadurai [1]

Department of Civil Engineering and Applied Mechanics, McGill University, Montréal, QC, Canada H3A 2K6

ARTICLE INFO

Article history:
Available online 11 October 2008

Communicated by K.R. Rajagopal

Keywords:
Reinforced elastic halfspace
Inextensible membrane reinforcement
Mixed boundary value problem
Dual integral equations
Fredholm integral equations
Surface indentation
Boussinesq–Harding–Sneddon indentation

ABSTRACT

The paper examines the problem of the axisymmetric Boussinesq–Sneddon–Harding indentation problem for an isotropic elastic halfspace, which is reinforced with an inextensible membrane that is placed at a finite depth from the surface. The resulting mixed boundary value problem is reduced to the solution of a Fredholm integral equation of the second-kind that is solved numerically. The numerical solution of the integral equation provides results that illustrate the influence of the depth of embedment of the reinforcing membrane and Poisson's ratio of the elastic material on the stiffness of the indentor.

1. Introduction

The class of problems that deal with the mathematical modelling of elastic regions reinforced with arrays of closely spaced inextensible reinforcing elements can be traced back to a classical study presented by Westergaard [1]. Other expositions of constrained media have been given by Wieghardt [2] and Vlazov [3]. Constrained continua also form the basis of idealized foundation models that have been used extensively to examine the interaction between geomaterials and structural elements (Selvadurai [4]). Modern continuum treatments of non-linear elastic media with inextensibility constraints were presented by Adkins and Rivlin [5] and Green and Adkins [6] in the study of rubber-like elastic materials reinforced with inextensible fibres. The treatise by Spencer [7] is a comprehensive study of the continuum theory of elastic materials reinforced with families of inextensible fibres. Further reviews of the topic are given by Spencer [8], Spencer and Soldatos [9].

The problem examined in this paper is prompted by an application where reinforcing layers are used to enhance the stiffness and strength characteristics of geomaterials. There is ongoing interest in modelling the mechanics of reinforced geomaterials to evaluate the efficacy of the reinforcement on the deformability and load carrying capacity of the reinforced geomaterial. In this paper we examine the *axisymmetric* problem of the surface indentation of an isotropic elastic halfspace, which contains an inextensible membrane-type reinforcement located at a finite depth below its surface. The paper extends the classical axisymmetric problem of the smooth indentation of an *isotropic elastic halfspace by a flat circular indentor* first examined by Boussinesq [10] and subsequently by Harding and Sneddon [11] to include the effect of an inextensible reinforcing layer. The *a priori* imposition of an inextensibility constraint in the entire plane of the reinforcing membrane is rec-

[1] William Scott Professor and James McGill Professor.
E-mail address: patrick.selvadurai@mcgill.ca

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doi:10.1016/j.ijengsci.2008.08.004

ognized as a limitation in the modelling; nonetheless, it provides, as does the use of the classical theory of elasticity (Davis and Selvadurai [12], Selvadurai [13]), a useful first approximation for the study of this class of problems. In reality, however, the inextensibility constraint will be meaningful only in regions where the membrane exhibits tensile in-plane forces. The alternative to this assumption is to consider a membrane with a *unilateral inextensibility constraint*, where inextensibility will materialize only when the membrane region experiences tensile tractions in its plane. The resulting problem involves a much more complicated formulation, where the region of applicability of the constraint is in itself an unknown in the problem. Such a problem is more conveniently formulated in an incremental computational setting where the non-linear effects of a moving inextensibility constraint over a circular region can be handled more conveniently. With incremental treatments, it is also possible to consider phenomena such as Coulomb friction and slip between the membrane and the surrounding elastic medium. In terms of an engineering treatment, which examines the influence of the inextensibility constraint on the indentor displacement, the bilateral inextensibility constraint acting over the entire region is considered to be a useful first approximation. Also, the consideration of an inextensibility constraint within the halfspace region assumes that the membrane is placed on the surface of a halfspace region and material is added to construct the layer region above the membrane. This is a practice that is often adopted in geotechnical constructions where the settlements of foundations are to be minimized, although the membrane reinforcement is not restricted to a single layer.

The general axisymmetric contact problem associated with the surface indentation of a halfspace reinforced internally by an inextensible membrane is formulated by considering a Hankel transform development of the governing equations and the reduction of the system of dual integral equations governing the mixed boundary value problem to a single Fredholm integral equation of the second-kind for an unknown function. The governing Fredholm integral equation of the second-kind is solved numerically using a quadrature technique to develop results of engineering interest. In particular, the numerical results illustrate the influence of the depth of embedment of the reinforcing layer on the elastic stiffness of the indentor.

2. Governing equations

The formulation of the axisymmetric problem in isotropic elasticity uses the strain potential approach proposed by Love and the displacement function approaches proposed by Boussinesq et al. (see, e.g. Truesdell [14], Green and Zerna [15], Gurtin [16], Gladwell [17] and Barber [18]). Following Green and Zerna [15], the solution to the axisymmetric problem in classical elasticity, referred to the cylindrical polar coordinate system, can be expressed in terms of two harmonic functions $\varphi(r,z)$ and $\psi(r,z)$ which satisfy

$$\nabla^2\varphi(r,z) = 0; \quad \nabla^2\psi(r,z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The displacement and stress components referred to the cylindrical polar coordinate system can be expressed in terms of $\varphi(r,z)$ and $\psi(r,z)$ as follows:

$$u_r = \frac{\partial\varphi}{\partial r} - z\frac{\partial\psi}{\partial r}; \quad u_z = (3-4\nu)\psi + \frac{\partial\varphi}{\partial z} - z\frac{\partial\psi}{\partial z} \tag{3}$$

and the stress components relevant to the formulation of the mixed boundary value problem are given by

$$\sigma_{zz} = 2\mu\left[2(1-\nu)\frac{\partial\psi}{\partial z} + \frac{\partial^2\varphi}{\partial z^2} - z\frac{\partial^2\psi}{\partial z^2}\right] \tag{4}$$

$$\sigma_{rz} = 2\mu\left[(1-2\nu)\frac{\partial\psi}{\partial r} + \frac{\partial^2\varphi}{\partial r\partial z} - z\frac{\partial^2\psi}{\partial r\partial z}\right] \tag{5}$$

where μ is the linear elastic shear modulus and ν is Poisson's ratio.

3. The contact problem

We consider the problem of an isotropic elastic halfspace region containing a circular inextensible membrane of infinite extent located at a finite depth h from the surface of the halfspace. The surface of the halfspace region ($z = -h$) is subjected to indentation by a rigid circular punch with a flat smooth base and of radius b, which induces a rigid displacement (Fig. 1). We further assume that the membrane region exerts an inextensibility constraint over the entire region $r \in (0,a)$. The origin of coordinates is located at the center of the inextensible membrane. In order to formulate the mixed boundary value problem we designate the halfspace region below the inextensible membrane and occupying $r \in (0,\infty)$ and $z \in (0,\infty)$ by the superscript $^{(2)}$ and the layer region above the inextensible membrane, occupying $r \in (0,\infty)$ and $z \in (0,-h)$, by the superscript $^{(1)}$. The mixed boundary value problem associated with the membrane-reinforced halfspace problem is as follows: for the layer region

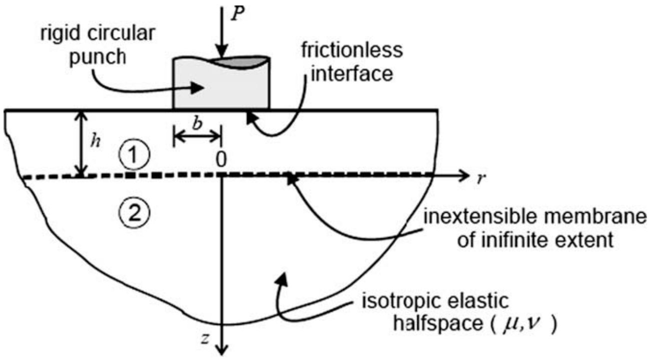

Fig. 1. The axisymmetric indentation of an elastic halfspace containing an inextensible circular membrane.

$$u_z^{(1)}(r,-h) = \Delta; \quad 0 \leqslant r < b \tag{6}$$
$$\sigma_{zz}^{(1)}(r,-h) = 0; \quad b < r < \infty \tag{7}$$
$$\sigma_{rz}^{(1)}(r,-h) = 0; \quad 0 < r < \infty \tag{8}$$

and at the plane of the inextensible membrane

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) = 0; \quad 0 \leqslant r < \infty \tag{9}$$
$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0); \quad 0 < r < \infty \tag{10}$$
$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0); \quad a < r < \infty \tag{11}$$

In addition, the displacements $u_i^{(1)}$ and stresses $\sigma_{ij}^{(1)}$ should satisfy the regularity conditions applicable to the halfspace region and the equilibrium equation for the rigid punch is given by

$$P + 2\pi \int_0^b \sigma_{zz}^{(1)}(r,-h)\mathrm{d}r = 0 \tag{12}$$

Considering a Hankel transform development of the governing partial differential Eq. (1) we obtain solutions separately applicable to the layer and halfspace regions [19]. For the halfspace region, we have

$$u_r^{(2)}(r,z) = \int_0^\infty [A(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_1(\xi r)\mathrm{d}\xi \tag{13}$$
$$u_z^{(2)}(r,z) = \int_0^\infty [A(\xi) + (3-4\nu)B(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_0(\xi r)\mathrm{d}\xi \tag{14}$$
$$\sigma_{zz}^{(2)}(r,z) = -2\mu \int_0^\infty \xi[A(\xi) + 2(1-\nu)B(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_0(\xi r)\mathrm{d}\xi \tag{15}$$
$$\sigma_{rz}^{(2)}(r,z) = -2\mu \int_0^\infty \xi[A(\xi) + (1-2\nu)B(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_1(\xi r)\mathrm{d}\xi \tag{16}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. For the layer region, the relevant expressions for the displacements and stresses are given by

$$u_r^{(1)}(r,z) = -\int_0^\infty \begin{bmatrix} \sinh[\xi(z+h)]\{C(\xi) - \xi z E(\xi)\} \\ + \cosh[\xi(z+h)]\{D(\xi) - \xi z F(\xi)\} \end{bmatrix} \frac{J_1(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \tag{17}$$

$$u_z^{(1)}(r,z) = \int_0^\infty \begin{bmatrix} (3-4\nu)\{E(\xi)\sinh[\xi(z+h)] + F(\xi)\cosh[\xi(z+h)]\} \\ +C(\xi)\cosh[\xi(z+h)] + D(\xi)\sinh[\xi(z+h)] \\ -\xi z\{E(\xi)\cosh[\xi(z+h)] + F(\xi)\sinh[\xi(z+h)]\} \end{bmatrix} \frac{J_0(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \tag{18}$$

$$\frac{\sigma_{zz}^{(1)}(r,z)}{2\mu} = \int_0^\infty \begin{bmatrix} 2(1-\nu)\xi\{E(\xi)\cosh[\xi(z+h)] + F(\xi)\sinh[\xi(z+h)]\} \\ +\xi\{C(\xi)\sinh[\xi(z+h)] + D(\xi)\cosh[\xi(z+h)]\} \\ -\xi^2 z\{E(\xi)\sinh[\xi(z+h)] + F(\xi)\cosh[\xi(z+h)]\} \end{bmatrix} \frac{J_0(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \tag{19}$$

$$\frac{\sigma_{rz}^{(1)}(r,z)}{2\mu} = \frac{\partial}{\partial r}\int_0^\infty \begin{bmatrix} (1-2\nu)\{E(\xi)\sinh[\xi(z+h)] + F(\xi)\cosh[\xi(z+h)]\} \\ +\{C(\xi)\cosh[\xi(z+h)] + D(\xi)\sinh[\xi(z+h)]\} \\ -\xi z\{E(\xi)\cosh[\xi(z+h)] + F(\xi)\sinh[\xi(z+h)]\} \end{bmatrix} \frac{J_0(\xi r)}{\sinh(\xi h)}\mathrm{d}\xi \tag{20}$$

where $C(\xi), D(\xi), E(\xi)$ and $F(\xi)$ are arbitrary functions. Considering the boundary condition (8) and the first of the boundary condition (9), we have

$$D(\xi) = -\tanh(\xi h)C(\xi); \quad C(\xi) = -\xi h E(\xi) - (1-2\nu)F(\xi) \tag{21}$$

and the second of the boundary conditions (9) gives

$$A(\xi) \equiv 0 \tag{22}$$

$$B(\xi) = \frac{1}{(3-4\nu)}\left[(3-4\nu)\{E(\xi) + F(\xi)\coth(\xi h)\} - \frac{\xi h E(\xi) + (1-2\nu)F(\xi)}{\sinh(\xi h)\cos(\xi h)}\right] \tag{23}$$

The vanishing of the shear stresses on the surface of the halfspace is

$$B(\xi) = -[F(\xi) + E(\xi)\coth(\xi h)] \tag{24}$$

which gives

$$E(\xi) = -\left(\frac{(3-4\nu)(1+\mathrm{e}^{-2\xi h}) - 2(1-2\nu)\mathrm{e}^{-2\xi h}}{(3-4\nu)(1+\mathrm{e}^{-2\xi h}) - 2\xi h\mathrm{e}^{-2\xi h}}\right)F(\xi) = \Psi(\xi)F(\xi) \tag{25}$$

We can express the displacement and normal stress on the surface of the halfspace region as follows:

$$u_z^{(1)}(r,-h) = 2(1-\nu)\int_0^\infty \frac{F(\xi)J_0(\xi r)\mathrm{d}\xi}{\sinh(\xi h)} \tag{26}$$

$$\frac{\sigma_{zz}^{(1)}(r,-h)}{2\mu} = \int_0^\infty \xi F(\xi)\left[\begin{array}{l}\xi h + 2(1-\nu)\Psi(\xi)\\ +\tanh(\xi h)\{1-2\nu+\xi h\Psi(\xi)\}\end{array}\right]\frac{J_0(\xi r)\mathrm{d}\xi}{\sinh(\xi h)} \tag{27}$$

and the boundary conditions (6) and (7) yield the following mixed boundary value problem for a single unknown function $\Phi(\xi)$:

$$\int_0^\infty \frac{\Phi(\xi)}{\xi}\eta(\xi,h)J_0(\xi r)\mathrm{d}\xi = \frac{\Delta}{2(1-\nu)}; \quad 0 \leqslant r \leqslant b \tag{28}$$

$$\int_0^\infty \Phi(\xi)J_0(\xi r)\mathrm{d}\xi = 0; \quad b < r < \infty \tag{29}$$

where

$$\Phi(\xi) = \frac{\xi F(\xi)}{\sinh(\xi h)}\Omega(\xi,h); \quad \eta(\xi,h) = \frac{2(1-\nu)}{\Omega(\xi,h)} \tag{30}$$

$$\Omega(\xi,h) = [\xi h + 2(1-\nu)\Psi(\xi) + \tanh(\xi h)\{1-2\nu+\xi h\Psi(\xi)\}] \tag{31}$$

Using the standard approach that employs a finite Fourier cosine transform [20] representation for $\Phi(\xi)$ of the form

$$\Phi(\xi) = \int_0^b \varphi(\xi)\cos(\xi t)\mathrm{d}t \tag{32}$$

the Eq. (29) is identically satisfied and (28) reduces to a Fredholm integral equation

$$\varphi(t) + \int_0^b K(u,t)\varphi(u)\mathrm{d}u = -\frac{\Delta}{\pi(1-\nu)}; \quad 0 < t < b \tag{33}$$

and the kernel function $K(u,t)$ is given by

$$K(u,t) = -\frac{1}{\pi(1-\nu)}\int_0^\infty [\eta(\xi,h) + 2(1-\nu)]\cos(\xi t)\cos(\xi t)\mathrm{d}\xi \tag{34}$$

4. Limiting cases and numerical results

The result of primary interest to engineering applications is the load (P)–displacement (Δ) relationship for the indentor and an evaluation of the influence that the depth of embedment of the inextensible flexible membrane has on the resulting load–displacement relationship. The load displacement relationship can be obtained from (12) and using the preceding results it can be shown that

$$P + 4\pi\mu\int_0^b \varphi(t)\mathrm{d}t = 0 \tag{35}$$

Owing to the complicated form of the kernel function (34), the Fredholm integral equation of the second-kind (33) does not appear to have an exact solution. There are, however, a number of numerical approaches that can be used to solve the Fredholm integral equation and these are documented by Baker [21], Delves and Mohamed [22] and Atkinson [23]. Two limiting results can be obtained quite conveniently from the results presented earlier: In the limit as $h \to \infty$, the reinforcing action of the inextensible layer has no influence and the indentation problem reduces to that of the classical result for the smooth indentation of a halfspace by a rigid circular punch with a flat base, where the load displacement relationship is given by

$$P = \frac{4\mu\Delta b}{(1-\nu)} \tag{36}$$

In the limit as $h \to 0$, the problem reduces to that of the indentation of a halfspace $(r \in (0,\infty); z \in (0,\infty))$ governed by the mixed boundary value problem

$$u_z^{(1)}(r,0) = \Delta; \quad 0 \leqslant r < b \tag{37}$$

$$\sigma_{zz}^{(1)}(r,0) = 0; \quad b < r < \infty \tag{38}$$

$$u_r^{(1)}(r,0) = 0; \quad 0 < r < \infty \tag{39}$$

The corresponding result for the load–displacement relationship is given by

$$P = \frac{16\mu\Delta b(1-\nu)}{(3-4\nu)} \tag{40}$$

This result can also be determined by making use of Kelvin's solution for the internal loading of an elastic infinite space by a concentrated force of magnitude $2P$, where, due to asymmetry, the plane $z = 0$, at the point of application of the concentrated force resembles an inextensible plane [18,24].

The numerical approach involves the conversion of the Fredholm integral equation of the second-kind to a matrix equation where the unknown function $\varphi(t)$ is expressed as a vector of a discrete number of unknowns. Considering the integral equation in a discretized form over N intervals, we can write (33) in the form

$$\varphi(t_i) + \frac{b}{N}\sum_{j=1}^{N} K(t_j, t_i)\varphi(t_j) = -\frac{\Delta}{\pi(1-\nu)} \tag{41}$$

where $t_i = (i-1)b/N; i = 1, 2, \ldots, N+1$. The discretized form of the result (35) is given by

$$P + \frac{4\pi\Delta\mu b}{(1-\nu)\Delta N}\sum_{i=1}^{N}\varphi(t_i) = 0 \tag{42}$$

The limiting estimates for the load–displacement relationship for the indentor given by (36) and (40), have effectively *bounded* the influence of the embedded inextensible membrane of infinite extent located at a finite depth below the surface of the halfspace region, applicable for any arbitrary value of Poisson's ratio ν. We also note that in the particular case of an incompressible elastic material, $\nu = 1/2$, and both (36) and (40) converge to the same result, indicating that in the case of *material incompressibility*, the provision of an inextensible flexible reinforcement at the surface of the halfspace does not contribute in any way to increasing the elastic stiffness of the circular indentor. This of course does not preclude the existence of a specific interior location of the inextensible membrane, which would increase the stiffness of the surface indentor. The numerical approach described previously is used to estimate the load–displacement relationship for the rigid circular indentor for the general case involving inextensible reinforcement of the elastic halfspace at an arbitrary depth from its surface. Fig. 2 illustrates the influence of Poisson's ratio (ν) and the depth of embedment (h) of the inextensible membrane. The results for the elastic stiffness of the rigid indentor applicable to the reinforced halfspace presented in Fig. 2 are normalized with respect to that for the unreinforced elastic halfspace. It is evident that the dominant effects of the inextensible membrane materialize when Poisson's ratio for the halfspace material is zero. In the case of an incompressible elastic material, the optimum location for the inextensible reinforcing membrane is at the interior of the halfspace region.

5. Concluding remarks

Reinforcement with a stiffer material is an approach used quite extensively to increase the stiffness characteristics of highly deformable constructed geomaterials. The mechanical behaviour of both geomaterials and the reinforcing materials, such as geosynthetics, are generally non-linear and rate-dependent. The linear elasticity model of the reinforcement of an elastic halfspace by an inextensible membrane presented in the paper is suitable for examining the influence of the reinforcing layer on the indentation stiffness of a rigid circular punch located at the surface of the halfspace region, since the objective of the reinforcement is to ensure that the deformations of the reinforced halfspace are not excessive. The analysis of the indentation of the halfspace reinforced with an inextensible membrane can be formulated as a mixed boundary value problem in elasticity, which can be reduced to the solution of the Fredholm integral equation of the second-kind. In particular, the indentation stiffness of the discretely reinforced halfspace can be obtained from a numerical solution of the integral equation. The stiffness is influenced by the Poisson's ratio of the elastic material and the depth of embedment of the reinforcing layer. The accuracy of the numerical estimates is also verified through limiting exact closed form solutions. The formulation presented here is also applicable to the reinforcement of the halfspace by sets of inextensible reinforcing fibres that are placed orthogonally to each other at a finite depth within the halfspace region. The assumptions of bilateral inextensibility can be eliminated by considering the inextensibility to be operative over a finite radius, which needs to be determined. The

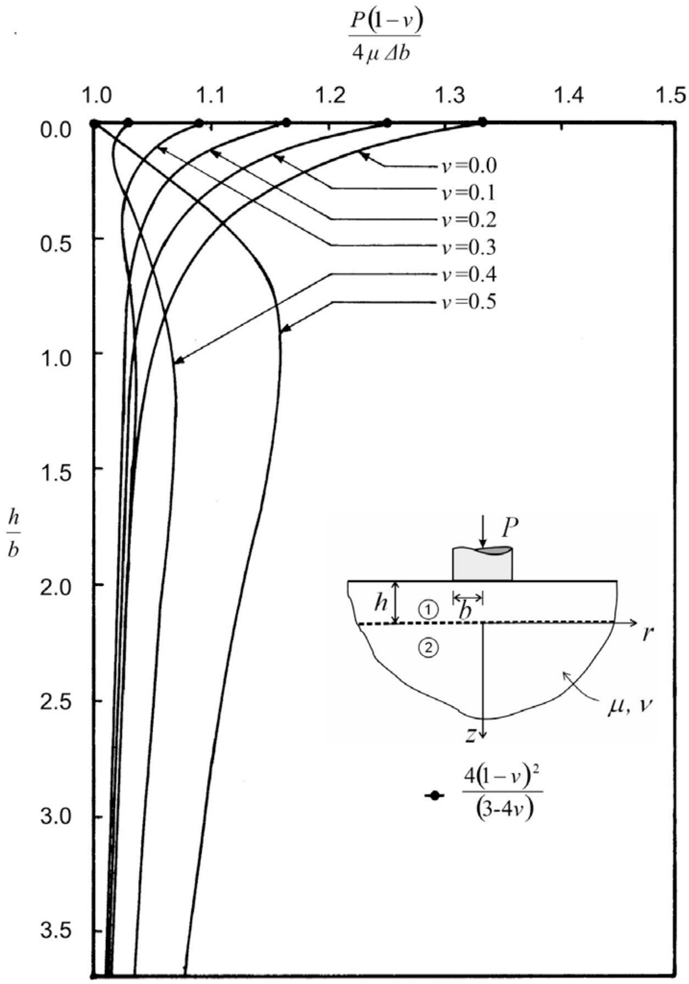

Fig. 2. Load–displacements response for a rigid indentor on an elastic halfspace containing an inextensible membrane.

complexity involved in the solution of the resulting mixed boundary problem cannot be justified in view of the elementary nature of the model and the scope of its application. The availability of computational approaches for solving elasticity problems means that more complicated configurations, particularly those involving layer regions of finite overall depth can be conveniently handled using such schemes. The computational modelling of the inextensibility constraint is, however, a different issue where the computational approaches *cannot* be directly applied without recourse to advanced formulations. If the *inextensibility constraint* is introduced through the incorporation of a *special element*, then the computational approach is entirely feasible. Also, if the *inextensible region* is of finite extent, which would be the objective of a computational approach, then special elements need to be incorporated to account for singular stress and displacement fields that will be present at the boundary of the reinforcing region. If an attempt is made to accommodate for the inextensibility constraint through prescribing a large elasticity mismatch between the membrane and the elastic medium and a thin layer "zero thickness" element, this will lead to ill-conditioning of the computational approach and to spurious deformation modes within the thin layer, which will make the computational approach of limited value.

Acknowledgements

The work reported in the paper was supported by the Max Planck Forschungspreis in Engineering Sciences awarded to the author. The comments of the reviewers are gratefully acknowledged.

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