PROSTHAPHAERESIS AND JOHANNES WERNER (1468 – 1522 [1])

-

A history of the forerunner of the logarithm and of its inventor.

$$\sin a \bullet \sin c = \frac{1}{2}\{\sin((90° - a) + c) - \sin((90° - c) - a)\}$$

(for 0 < a, 0 < c, 0 < a+c < 90°)

by

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[1] The German version [Kuehn 2008] states the year as 1528, which is not correct according to recent findings; Nuremberg 2009

1 Acknowledgements/Dedications

The authors thank Professor Dr. Menso Folkerts for information concerning a most important document about Prosthaphaeresis: [Björnbo]. Interestingly, this source has been digitised recently which might reflect the importance of this publication.

The authors also thank Ms. Barbara Häberlin, Mr. Stephan Drechsler and Mr. Stephan Weiss for their critical comments and suggestions for changes to the original German version of this work.

Furthermore, the authors thank Ms. Magister Dr. Gerlinde Faustmann for providing a reference to Jost Bürgi's work on the proof of the prosthaphaeretic formulae.

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2 Abstract

English
For most of his life Johannes Werner (1468-1522) was a priest and astronomer living in Nuremberg, Germany. He first published the prosthaphaeretic formulae (the term “prosthaphaeretic” coming from the Greek for addition and subtraction) around 1513 in a manuscript; this information is mainly supported by very intensive research carried out by Axel Anthon Björnbo (1874-1911) [Björnbo].

It is not exactly known if Werner was aware at that time of the advantageous use of the prosthaphaeretic formula for calculations with multidigit numbers; however, this can be assumed as being the case.

Moreover, strong evidence shows that neither the astronomer Tycho Brahe (1546-1601) nor his student Paul Wittich (1555? - 1587) invented the prosthaphaeretic formula. However, Tycho Brahe was among the first, who - from 1580 to 1601 - took intensive advantage of the prosthaphaeretic formula for his astronomical calculations.

This paper reviews the historical background for the formulation and "re-invention" of prosthaphaeresis.

On the basis of the relevant literature it gives some practical examples as well as the mathematical-geometrical proof of the formula.

Deutsch
Johannes Werner (1468-1522), Pfarrer und Astronom, verbrachte die meiste Zeit seines Lebens in Nürnberg und ist derjenige, der um 1513 die prosthaphäretische Formel - der Begriff kommt aus dem Griechischen und steht für Addition und Subtraktion (siehe Formel im Titel) - in einem Manuskript festhielt.

Besonders die gründliche Untersuchung von Axel Anthon Björnbo (1874-1911) [Björnbo] legt dazu inzwischen zahlreiche Hinweise vor. Ob Werner deren Eignung als Rechenmethode für die Multiplikation mehrstelliger Zahlen bewusst war, lässt sich nicht mit Bestimmtheit sagen, liegt aber nahe. Sicher ist, dass es weder der Astronom Tycho Brahe (1546-1601) noch sein Schüler Paul Wittich (1555? - 1587) waren, die diese Formel entdeckt hatten. Tycho Brahe war allerdings einer derjenigen, der die Prosthaphärese zu seiner Zeit - zwischen 1580 und 1601 - sehr intensiv zu (astronomischen) Berechnungen einsetzte.

In diesem Aufsatz werden die Hintergründe zur Darstellung und "Wiederentdeckung" der Prosthaphärese, deren Anwendungsbereiche und ein mathematisch-geometrischer Beweis der Formel auf Basis der relevanten Literatur aufgezeichnet.

3 Table of contents

4 Introduction

For a long time, people have looked for ways to simplify computing procedures. It was not so important how difficult the calculations might have been; the goal was always to reduce the amount of computation, but without losing any accuracy.

Particularly in the field of astronomy, in which mathematics first developed, where computations with multidigit numbers were (and still are) a necessity, these computations were very expensive in time and effort. This particularly concerned the basic operations of arithmetic such as multiplication and division, so if it would be possible to simplify these operations, for example by reducing multiplication to addition, then that would be an ideal solution.

The most well-known example of this methodology, the reduction of multiplication to simpler functions, is the logarithm; logarithms were published in 1614 in Edinburgh by John Napier (1550-1617) in the first table of logarithms (Mirifici Logarithmorum Canonis Descriptio).

But what happened before then? How did astronomers do their calculations without having logarithms at their disposal?

The answer is that for about hundred years they used ***prosthaphaeresis***, (also written as prosthaphairesis or prostaphairesis)

Literature on the subject of Prosthaphaeresis frequently mentions an incorrect name as its inventor; usually the discovery is attributed to the astronomer Tycho Brahe or to his pupil Paul Wittich, or sometimes even to Christopher Clavius. However, Brian Borchers gave a short overview of Prosthaphaeresis and its history in his article in the Journal of the Oughtred Society (JOS) [Borchers], and in that article he referred to its originator as being Johannes Werner.

Borchers' article stands as the starting point for this article, in which the background to the Prosthaphaeretic formula, and to Prosthaphaeresis itself, will be clarified from historical and mathematical viewpoints.

The term “Prosthaphaeresis” - meaning a system in which one uses addition and subtraction - also has other usages in astronomy; thus one speaks, for example, of prosthaphaeresis in connection with: aequinoctiorum; eccentritatis; latitudinis; nodi pro eclipsius; orbis; tychonica; nodorum - “an orbiting body does not move itself evenly; it moves more slowly if the Sun is in the proximity of the body; faster, if the Sun moves away from it. “ [Bialas]. However, these purely astronomical usages of the term "prosthaphaeresis" will not be considered further in this essay.

Johannes Werner (1468 - 1522) can be seen to be the discoverer of Prosthaphaeresis, and substantial support for this can be found in a work by Axel Anthon Björnbo [Björnbo]. As a disciple of the historian of science Anton von Braunmühl, Björnbo took up von Braunmühl's references to some inconsistencies and went to Rome in 1901, so that he could read and study appropriate ancient material in the Vatican library.

Of particular interest, he found an undated manuscript, with the title: ' I. Ioannis Verneri Norimbergensis "de triangulis sphaericis [2]" ' in four books, and also "II. Ioannis Verneri Norimbergensis "de meteoroscopiis" ' in six books.

Queen Christina of Sweden had been in possession of this manuscript, probably between 1654 and 1689; this document had been previously owned by Jakob Christmann (1554 - 1613).

[2] For easier reading, this word is spelled in this article "sphaericis" instead of the original "sphoericis" (see Figure 5-3)

After Queen Christina’s death in 1689, this manuscript (Codex Reginensis latinus 1259, i.e. Regina Sveciae Collection, item 1259) lay mainly disregarded in the Vatican. The journey of this manuscript through time is documented in chapter 11.

During further investigations it became clear that Werner was the editor and/or an author of the two handwritten parts, but that he did not physically write them himself.

As to the actual writer of the document, Björnbo identified a mathematically-inexperienced professional scribe of the time.[Björnbo; Pages 140, 141, 171].

The text of the first complete part of the manuscript (*de triangulis sphaericis*) can be found in Björnbo's work [Björnbo; Chapter 1] on pages 1 - 133. Later on Björnbo voices his opinions concerning this manuscript both in the “publisher remarks” [Björnbo; Chapter 3] and also in “text history” [Björnbo; Chapter 4] in a very detailed research report.

However, the extensive contents of that manuscript will not be dealt with further in this essay.

It should be mentioned however, that as an innovation for its time, the organisation/arrangement of the books concerning spherical triangles is the following [Björnbo; Chapter 1 and pages 163; 177-183]:

1. **An explanation of the different possible triangle forms (Book I)**
    a. **A discussion concerning the spherical triangle**

2. **Solutions of the right-angled triangle (Book II)**
    a. **The spherical-trigonometric basic formulae**
    b. **The solution of the right-angled spherical triangle**

3. **Solutions of the non-right-angled (obtuse) triangle (Book III and IV)**
    a. **The solution of the obtuse triangle by decomposition into right-angled triangles (Book III)**
    b. **The solution of the obtuse spherical triangle by a prosthaphaeretically transformed Cosine rule (Book IV)**

The three categories stated are of the same structure as the contents of the *Opus Palatinum* of Rheticus of 1596, in so far as spherical triangles are concerned.

In chapter 5 [Björnbo; starting from page 177] is summarised the structure of the contents of the individual books in a tabular form.

Thus the findings of Björnbo, namely, the assumptions of Anton von Braunmühl [von Braunmühl 1897] which confirm the authorship of the prosthaphaeretic formula, along with recent up-to-date work by David A. King [King] and Victor E. Thoren [Thoren] together constitute the foundation for the remainder of this article.

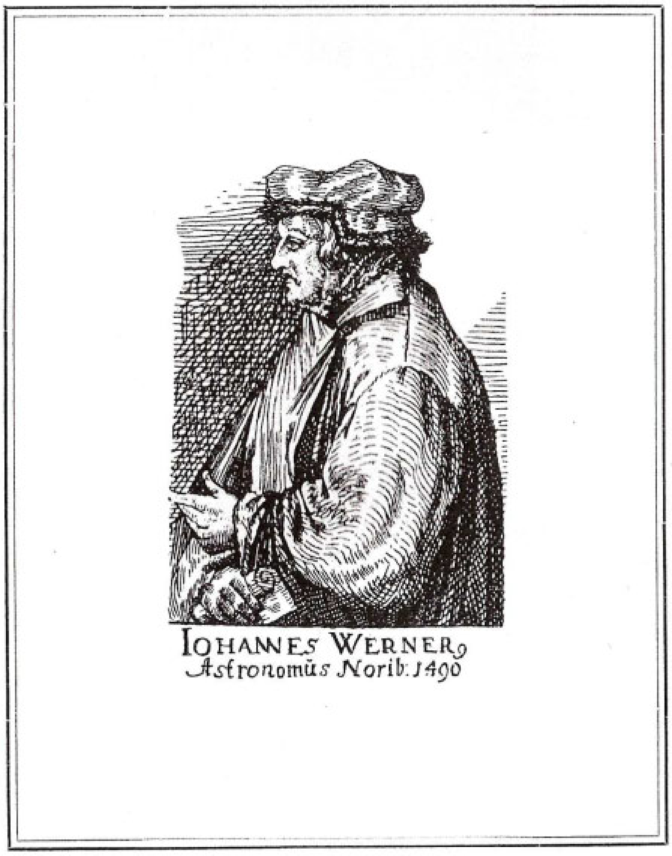
5 Historical

The history of Prosthaphaeresis is summarised in the time table in the appendix, and is derived from several sources [Björnbo; von Braunmühl].

Here now it is necessary to briefly describe the details of Johannes Werner's life, along with the steps in time of the development of Prosthaphaeresis, including its "rediscovery" and its sequence of publication.

Johannes Werner was born on 14 February 1468 in Nuremberg and died in (May?) 1522 in Nuremberg while he was in the position of parish priest in the municipality of St. Johannes.

In his spare time he worked as a mathematician, astronomer, astrologer, geographer and cartographer.



Werner studied theology and mathematics in Ingolstadt from 1484 onwards. In 1490 he became a chaplain in Herzogenaurach.

From 1493 to 1497 he was in Rome.

In 1503 he was appointed as the vicar at the church in Wöhrd, a suburb of Nuremberg.

Afterwards he became a priest at the Johanniskirche in Nuremberg; he held this position up to his death.

Kaiser Maximilian I. appointed him the imperial Chaplain.

Astronomers honoured him by naming a crater on the Moon as "Werner" (see Wikipedia).

Figure 5-1 Johannes Werner

Werner was very much interested in Astrology and created horoscopes for numerous well-known Nuremburg residents, including Erasmus Topler (1462-1512), Provost of St. Sebald, Willibald Pirckheimer (1470-1530), Christoph Scheurl II (1481-1542) and Sebald Schreyer (1446-1520). However, Werner gained harsh criticism from these activities. Lorenz Beheim (around 1457-1521), a choirmaster in Bamberg, wrote about him thus: "He always makes a big thing of his secrets, which however result in little honour for him. Mostly, if he wants to predict the truth, he invents it.

Werner became friendly with Johannes Stabius (approximately 1460-1522). In co-operation with him, he developed numerous important works. Werner suggested the construction of a sun-dial, designed to show "Nuremberg time", which essentially meant that the clock should indicate the hours passed since sunrise. Stabius supplied a design, which Sebastian Sperantius (? - 1525) drew on the east choir of the Lorenzkirche in 1502.

Stabius also pushed Werner to publish his manuscripts. In November 1514, the compilation under the management of Conrad Heinfogel (? - 1517) left the printing press. Amongst other things therein, a certain form of the map projection is presented, which is known to historians as the Stabius-Werner Projection. In 1522 appeared a second compilation (Figure 5-2), which contained his work, "On the Motion of the Eighth Sphere" or "De motu octavae Sphaerae". He studied the precession of the stars from the geocentric point of view; however, for this he was fiercely criticised by Copernicus (1473 – 1543) in the latter's "Letter against Werner"[Sobel].

This and other information, particularly also concerning Werner's meteorological activities, can be found on the Internet under [Nuremberg] and [Wikipedia].

A first compilation was published in 1514 under the title: "In hoc opere haec continentur: Nova translatio primi libri geographiae Cl. Ptolemaei, quae quidem translatio verbum habet e verbo fideliter expressum Ioanne Vernero Nurembergensi interprete……", containing work by him and by other authors.

In hoc opere haec continentur.

LIBELLVS IOANNIS VERNERI NVREMBERGEN. SVPER VIGINTIDVOBVS ELEMENTIS CONICIS.

EIVSDEM Commentarius seu paraphrastica enarratio in undecim modos conficiendi eius Problematis quod Cubi duplicatio dicitur.

EIVSDEM Commentatio in Dionysodori problema, quo data sphaera plano sub data secatur ratione,

ALIVS modus idem problema conficiendi ab eodem Ioanne Vernero nouissime compertus, demonstratusque.

EIVSDEM Ioannis, de motu octauae Sphaerae, Tractatus duo.

EIVSDEM, Summaria enarratio Theoricae motus octauae Sphaerae.

Cum Gratia & Priuilegio Imperiali.

From that compilation and from his own publications, what we know of Werner's life is only the following:

Starting before 1513, but probably after 1505, Werner wrote five books concerning spherical triangles with the title "Liber de triangulis sphaericis" or "Liber sphaeraliurn triangulorum".

During the years 1514 to 1522 this work underwent editing and collation.

Werner was very eager to have the work published, particularly because he was very aware as early as 1514 that the prosthaphaeretic method had a great value. [Björnbo, P. 157]

Figure 5-2 Compilation of 1522 [according to Mehl - from the library in Lisbon]

Björnbo draws this conclusion from the similarity of the contents of the manuscript with the contents of the compilations. On the one hand it concerned the amazing similarity and/or sameness of the solution of a triangle using orthogonal projection. Therein Werner had "already written in a pure form the prosthaphaeretic method and its application for the practical transformation of the Cosine Rule, i.e. the second main rule of spherical Trigonometry*."

*

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}$$

Further he says [Björnbo; Page 155]: “In the compilation of the year 1522 there appears in Werner's book "De motu octavae Sphaerae"…. in the triangle (star[3]***;*** pole of the ecliptic; north pole) the height of the Star (λ) by its width (ß), its declination (δ) and its inclination to the ecliptic (ε), i.e. that the angle of a skew spherical triangle is numerically determined by its three given sides….“

Björnbo takes the fact that the emergence of prosthaphaeresis must have taken place after 1505 from one of the available quotations in the translation of Euclid's work by Zamberti (Bartholomäo Zamberto Veneto) which became available only after 1505.

[3] The triangle is determined by the three corners: star; pole of the ecliptic; north pole

After it was clear that the manuscript Cod. Reg. 1259 had its origin in these two works by Johannes Werner, the search was on, after Werner's death, to find out the development and the whereabouts of this Cod. Reg. 1259.

Up to the death of Werner in the year 1522 the two works had still not been printed, or at least, no appropriate references or copies have been found from that time.

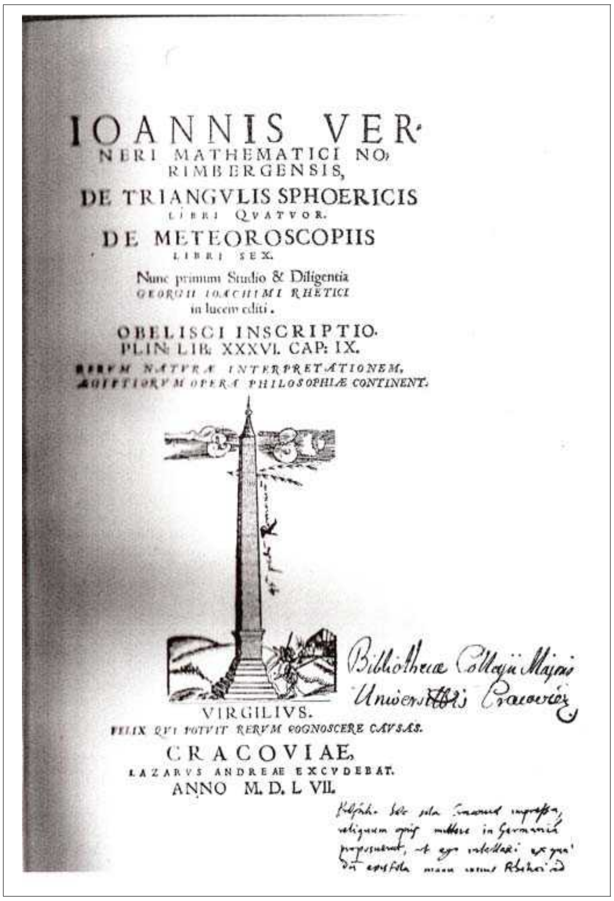
The contents of Book I - Ioannis Verneri *Norimbergensis "De triangulis sphaericis"* in four books, as well as Book II - Ioannis Verneri *Norimbergensis "De meteoroscopiis"* in six books, were however well known to Werner's contemporaries, including Johann Wilhelm von Loubemberg and his colleague Peter Apian.

The bibliographer Konrad Gesner (1516-1576) describes in 1555 that the Nuremberg mathematician and mechanic George Hartmann (1489-1564) saved the two works of Werner from destruction. According to Doppelmayr, Hartmann probably handed over these and other works from Werner's estate in 1542 in Nuremberg to George Joachim Rheticus (1514-1576; who lived from 1554 in Kraków as a practicing physician).

G. Eneström [Eneström] determined that both works of Johannes Werner were published by Rheticus in the year 1557 in Kraków.

However this publication contained, apart from the title page, only the ten-page introduction (The Prooemium) by Rheticus and nothing else which Werner wrote.

The title page contains clear references to the titles of the two books mentioned above.



IOANNIS VERNERI MATHEMATICI NORIMBERGENSIS,

DE TRIANGVLIS SPHOERICIS LIBRI QVATVOR.

DE METEOROSCOPIIS LIBRI SEX.

Nunc primum Studio & Diligentia GEORGII IOACHIMI RHETICI in lucem editi.

OBELISCI INSCRIPTIO. PLIN. LIB. XXXVI. CAP. IX.

RERVM NATVRAE INTERPRETATIONEM, AEGYPTIORVM OPERA PHILOSOPHIAE CONTINENT.

VIRGILIVS.

FELIX QVI POTVIT RERVM COGNOSCERE CAVSAS.

CRACOVIAE, LAZARVS ANDREAE EXCVDEBAT. ANNO M. D. L VII.

Bibliotheca Collegii Majoris Universitatis Cracoviensis

Figure 5-3 The title page of the Kraków publication.

Björnbo sees, as an explanation for the absence of the text, the fact that Rheticus, and after his death his disciple Valentinus Otho (approx. 1550-1605), had incorporated both the arrangement (a systematic presentation of different triangle forms) and the contents of the "De triangulis sphaericis", in the great book of tables *Opus Palatinum* (which was published in 1596) and they had perfected the philosophies of Johannes Werner, whom they both admired and respected.

However Rheticus did not himself support the solution of spherical triangles either by Ptolemaios' method or by Geber's method (which was developed by Peurbach, Regiomontanus and Werner).

So, he developed his own method independently; this method derived from the geometry of pyramids, using common points at the centre of the sphere; this latter methodology is derived from Copernicus [Björnbo, page 163 foot-note 2].

Rheticus was the only disciple of Copernicus and by his publication of the famous "De revolutionibus orbium coelestium Libri VI" had himself taken up the cause of providing a reliable sine table.

Thus Björnbo assumes the Vatican manuscript Cod Reg 1259 was in Rheticus' possession and represented a copy of the original, and that it should serve as a beginning point.

This printed manuscript - which contained no drawings - fell into the hands of his disciple Valentinus Otho after the death of Rheticus in 1576.

From this bequest the manuscript went to the Heidelberg professor Jakob Christmann (1554-1613) {Björnbo P. 165 incorrectly describes the date of death as 1630}, who quoted from the two works of Werner in his book *"Theoria lunae"* (1611), and even indicated that he possessed the two books.

In his dissertation of 1924, Erwin Christmann (a descendant of Jakob Christmann) wrote [Christmann]:

- *" The "Theoria lunae" plays a remarkable role in the history of trigonometry, as it gave in an appendix, information concerning the inventor of the prosthaphaeretic method.*

  Until the discovery of Werner's two documents, "de triangulis sphaericis" and "de meteoroscopiis" by A. Björnbo in 1902 in the Vatican library in Rome, the "Theoria lunae" was one of the few sources to bring clarity over this long disputed question.

  For von Braunmühl in 1899 in his "Lectures on the History of Trigonometry", Christmann's writing is the most outstanding support for his proof of the invention of the prosthaphaeretic method by Johannes Werner.

  Christmann explained here, that the manuscript of that work was well-known to him, -although it is not known whether it was the original manuscript, later lost, or the printed copy from the Vatican library, which was available to him -. Werner developed and described in figures the Prosthaphaeresis. He defended this against Tycho Brahe, who with his disciple Wittich, was generally regarded as being the inventor. Christmann is probably referring to a transcript, which would be good as a basis on which to work; his words therefore do not suggest a deliberate deception.

- *"Even today these relationships are not as clear as could be desired. It is feasible to recognise Werner as the inventor of the method and as the person who saw the opportunities for its possible use; however, in reality he is more the <u>re-discoverer</u> of the prosthaphaeretic formulae, as they were already well known to Arabic mathematicians. On the other hand, one must be objective and the trustworthy mathematical and astronomical circle of Count Wilhelm of Hessen above all ascribe to Wittich and Tycho Brahe the exclusive merit of the general introduction of the use of the prosthaphaeretic formulae in calculation. The meaning of their activity must be recognised all the more, in that the holy-of-holies inventors of logarithms and of their practical use had not become available. Furthermore, that this was not a collection of formulae by Wittich and Tycho Brahe can be proven by comparative research.*

- *In addition to the information given in the "theoria lunae" Christmann brings a full development of the method and key phrases from the triangle theory, so far as it was required.*

  He had already summarised these into his works "observationum solarium libri tres, in quibus explicatur versus motus Solis in sodiaca et universa doctrina triangulorum ad rationes *apparentius coelestium accomodatur Basel 1601". In another work called "nodus Cordinis ex doctrina sinum explicatus 1612" he taught the solution of geometrical problems with the help of sines, instead of using algebraic methods.*

- *Although today, by the rediscovery of Werner's trigonometrical work, the "theoria lunae" with its data has receded into the background, nevertheless its existence remains historically notable, particularly because its statements were, as a result of recent investigations, accepted as correct and also because together with the two writings from the years 1601 and 1612 written by a professor from Heidelberg interested in trigonometry, it puts down a clear testimony."*

Anton von Braunmühl based his remarks for the development of the prosthaphaeresis particularly on the statements of Jakob Christmann. He sees the origin of these formulae as being with Ibn Yunus, an Arab mathematician who died in 1009. However, according to David A. King [King], on the basis of new knowledge which he acquired while working on his thesis, this idea is no longer valid.

What role does Tycho Brahe (1546-1601), the Danish astronomer, play in connection with Prosthaphaeresis, which he himself began to use in 1580?

According to [von Braunmühl 1899] *"Tycho Brahe knew the source, in which Werner, using his trigonometrical books, applies the prosthaphaeretic method in order to find the elevation of Spica Virginis, because he often speaks of Werner´s writing "De motu octavae Sphaerae" and he (Tycho) particularly drew upon this observation of Spica. However the wording of that source could make it attentive only of the existence of a more practical calculation procedure than the usual one is, the procedure itself was absolutely not taken out of that source."*

It is possible that Brahe either had direct access to Johannes Werner's manuscript, or that it may be assumed that the manuscript's contents were known to him. [Björnbo, Page 168 ff] There are several ways in which this might have happened; see also [Thoren]:

1. During Brahe's visits to Wittenberg in the years 1566 or 1568-1569 or 1575, he may have seen Johannes Werner's books about triangles.

2. Paul Wittich and Brahe could have developed their own prosthaphaeretic method in 1580.

3. Reimarus Ursus (Nicolai Reimers; 1551-1600) - during a visit to the island Hven, where Brahe worked in 1584, may have stolen the prosthaphaeretic formula, and was thereafter considered as an intimate enemy of Brahe. In Ursus` *Fundamentum Astronomicum* (Strasbourg 1588) Johannes Werner's prosthaphaeretic formula is published for the first time.

4. Jost Bürgi, who was in contact with Wittich, may have played a role and may have received the formula from Wittich in Kassel - according to [Thoren] and [Lutstorf], Bürgi may then have provided the geometrical proofs.

5. Johann Richter (also known as Praetorius) (1537-1616) saw the book concerning spheres in 1569 written by Rheticus (he writes about it in 1599) and was from 1571-1576 a Professor of Mathematics in Wittenberg. According to a letter which Brahe wrote in 1588 to Hayck, he had not met Praetorius in 1575.

6. The role of Paul Wittich - to whom Brahe in 1592 (5 years after Wittich's death) ascribed the discovery of the Prosthaphaeresis. This is also proposed by [Thoren], who differentiates between the prosthaphaeretic formula itself, and actual computations with that formula.

Possibly it was a mixture of the above points, which led to the fact that Tycho Brahe became acquainted with prosthaphaeresis and then further developed it with Paul Wittich and learned how to use it. Anton von Braunmühl [von Braunmühl; Part 1, page 193] therefore speaks of a "re-invention" of prosthaphaeresis by Brahe in the year 1580. *Also Kepler refers to prosthaphaeresis on one occasion as "Artificium Tychonicum", then again as "Negotium Wittichianum" and finally as "Regula Wittichiana"* [von Braunmühl 1899].

The historical journey of the manuscript and of the formulae are graphically summarised in the appendix.

Now, the significance of Regiomontanus (Johannes Mueller, born in 1436 in Königsberg near Hassfurt - died in 1476 in Rome) concerning the work of Johannes Werner will be considered. Björnbo [Björnbo, page 172ff] explains the fact that Werner gained access to Regiomontanus' works, among other things the five "unfinished and mutilated" triangle books quite late - in fact, as late as 1504. Werner was not happy about this, and perhaps for this reason makes no reference in his own work to Regiomontanus, and does not cite the latter's work. Perhaps, because he was very familiar with the works of Euclid, Menelaus, Geber, Ptolemaios and Peurbach as used by Regiomontanus, he did not want to repeat Regiomontanus' work. However similarities can be seen in the ideas and in some of the expressions found in Regiomontanus' work and in Werner's work.

In connection with the history of prosthaphaeresis there frequently appear names of some very well-known and of some less known scholars, who cannot be dealt with in great detail here, but who should not be completely ignored. Their roles and work in connection with prosthaphaeresis are probably worth a completely separate investigation, but their names and some details are given here:

- Jost Bürgi (1552 - 1632)
- Peter Apian (1495 - 1552)
- Erasmus Reinhold (1511 - 1553)
- Bartholomäus Scultetus / Schulz (1532 - 1614)
- Christoph Clavius (1537 - 1602); (1538 - 1612, is also mentioned as the inventor of the Prosthaphaeresis [symposium 2005]) – he is not, however.
- Nicolaus Reimers /Reimarus Ursus (1551 - 1600)
- Paul Wittich (1555 - 1587)
- Melchior Jöstel (1559 - 1611) and his handwritten treatise *"Logistica Prosthaphaeresis Astronomica"* which can be found in the library of the Austrian National Library, Vienna (Cod. palat. 10686-27) [von Braunmühl 1899] as well as in the Dresden Landesbibliothek (# C 82; private communication from Menso Folkerts).
- Christian Severin Longomontan (1562 - 1647) and finally
- Ibn Yunus (around 1000).

Apart from the first and last, the above names are chronologically ordered according to their years of birth.

Much introductory information and references to these above people can be found in [von Braunmühl 1900], [Lutstorf] and [Thoren] and also in [Gingerich 1988] and [Gingerich 2005].

6 Mathematical

To remind the reader, prosthaphaeresis provides a method by the means of which the process of multiplication can be converted into an addition or a subtraction by the use of trigonometric formulae. This technique provided a substantial easing of work for the astronomers of the time.

Looking at books of formulae or on the Internet [Mathworld 11 and 12] it becomes clear that there are many ways of expressing the Prosthaphaeresis formulae. These formulae, which we know as the "prosthaphaeretic formulae" or as the "prosthaphaeresis formulae", are also known as the “Werner Formulae” or as the “Werner Formulas” (Figure 6-1).

$$
\begin{array}{lcll}
2\sin\alpha\cos\beta & = & \sin(\alpha-\beta)+\sin(\alpha+\beta) & (1)\\
2\cos\alpha\cos\beta & = & \cos(\alpha-\beta)+\cos(\alpha+\beta) & (2)\\
2\cos\alpha\sin\beta & = & \sin(\alpha+\beta)-\sin(\alpha-\beta) & (3)\\
2\sin\alpha\sin\beta & = & \cos(\alpha-\beta)-\cos(\alpha+\beta). & (4)
\end{array}
$$

Figure 6-1 The Werner Formulae

The URL on the above website leads to the Prosthaphaeresis Formulae as shown below in Figure 6-2 and which are known as "Simpson's Formulae" or "Simpson's Formulas". However these formulae differ in their representation and in their ease of use.

$$
\begin{array}{lcll}
\sin\alpha+\sin\beta & = & 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right] & (1)\\
\sin\alpha-\sin\beta & = & 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right] & (2)\\
\cos\alpha+\cos\beta & = & 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right] & (3)\\
\cos\alpha-\cos\beta & = & -2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right]. & (4)
\end{array}
$$

Figure 6-2 Prosthaphaeresis Formulae from Mathworld.

(In German linguistic usage [von Braunmühl 1900] these formulae shown in Figure 6-1 are known as "Die prosthaphäretischen Formeln".)

In more modern collections of formulae these names are not used, but instead the formulae are referred to as “products of trigonometric functions” [Bartsch] - see Figure 6-3.

$$
\begin{aligned}
\sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]\\
\cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]\\
\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\
\cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]
\end{aligned}
$$

Figure 6-3 The Prosthaphaeretic formulae as “products of trigonometric functions” = Figure 6-1.

The formulae used by Johannes Werner and Tycho Brahe and/or Paul Wittich differ from each other and therefore suggest a different, possibly independent, line of development. Whereas Werner works with the more elegant *Sinus Versus* (see Formula 1) Brahe only uses the sine function (see Formula 2) - [Björnbo, page 169].

Formula 1:

$$\frac{\dfrac{\sin(90° - a + c) - \sin(90° - a - c)}{2}}{\sin(90° - b) - \sin(90° - a - c)} = \frac{r}{\sin vers(180° - B)}$$

with r = 1; a; b; c = angles; B = side of the triangle

Formula 2:

$$\frac{\dfrac{\sin(90° - a + c) - \sin(90° - a - c)}{2}}{\sin(90° - b) - \dfrac{\sin(90° - a + c) - \sin(90° - a - c)}{2}} = \frac{r}{\sin(90° - B)}$$

with r = 1; a; b; c = angles; B = side of the triangle

By reformulating Formula 1 [Björnbo; Page 166] Werner finally arrives at Formula 3 (for a+c < 90°) and at Formula 4 (for a+c > 90°). For these, he used the sine form; he did not use cosine functions.

Formula 3:

$$\sin a \bullet \sin c = \frac{1}{2}\{\sin((90° - a) + c) - \sin((90° - c) - a)\}$$

Formula 4:

$$\sin a \bullet \sin c = \frac{1}{2}\{\sin((90° - a) + c) + \sin(a - (90° - c)\}$$

with a; c = angles

For the geometrical proof of the prosthaphaeretic formulae we go back to the remarks of Anton von Braunmühl [von Braunmühl 1900, page 195].

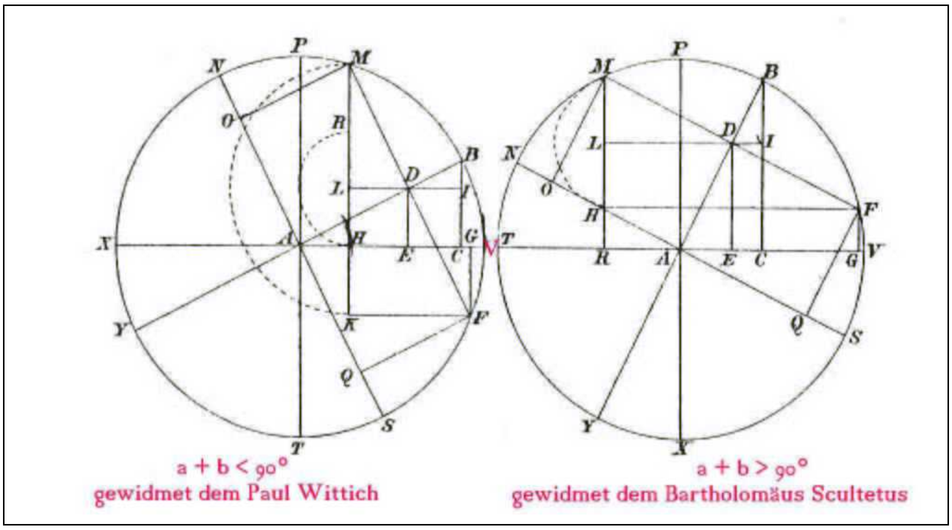


Figure 6-4 from Nikolaus Raymarus Ursus "Fundamentum astronomicum" 1588 (dedicated to Paul Wittich and Bartholomäus Sculteus)

Ursus formulated the two acknowledgments above.

In both figures arc BV = a, and arc SF = arc MN = b.   FDM is perpendicular to AB.

BC, DE, FG and MH are perpendiculars dropped onto AV.

In the left picture, HK = FG = MR, in the right HR = FG, and in both cases MO and FQ are perpendiculars dropped on NS, which runs parallel to MF.

In the left picture, with DLJ parallel to AV, arc VM = 90° - b + a and arc VF = 90° - b - a;

MH = sin (90° - b + a); FG = HK = MR =sin (90° - b - a).

With LH = DE = ½ RH = ½ (MH - MR) = ½ {sin (90° - b + a) - sin (90° - b - a)}.

Furthermore BC = sin a and AD = FQ = sin b and also AB: AD = BC: DE and also AB = Sinus totus (Sintot)

$$\text{Sintot: } \sin b = \sin a : \tfrac{1}{2} \{\sin (90° - b + a) - \sin (90° - b - a)\}$$

and from the second figure follows similarly:

$$\text{Sintot: } \cos b = \cos a : \tfrac{1}{2} \{\sin (b + 90° - a) - \sin (b - 90° + a)\}$$

In the following we present another geometrical proof from the work of Anton
von Braunmühl [von Braunmühl 1897, page 26 and von Braunmühl 1900, page 39, 63].

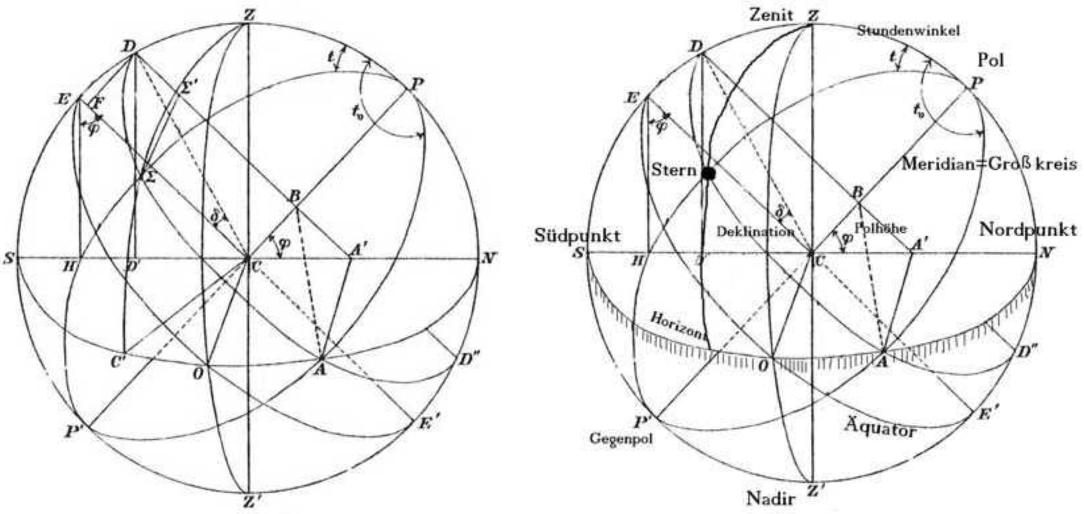


Figure 6-5 schematically drawn sphere with the axes of the horizontal and equatorial coordinates (ZZ' and PP') - the right figure shows a simplified version, with labels added.

In Figure 6-5 we recognise two angles which play an important role in astronomy:
Angle ∢ DCE = δ = declination and angle ∢ PCN = φ = height of the pole.

On the basis of these two angles the prosthaphaeretic formula can be deduced, if the diagram of the sphere is orthogonally projected onto the meridian SZN; this is a method which Arabic mathematicians used during their astronomical investigations. See Figure 6-6.

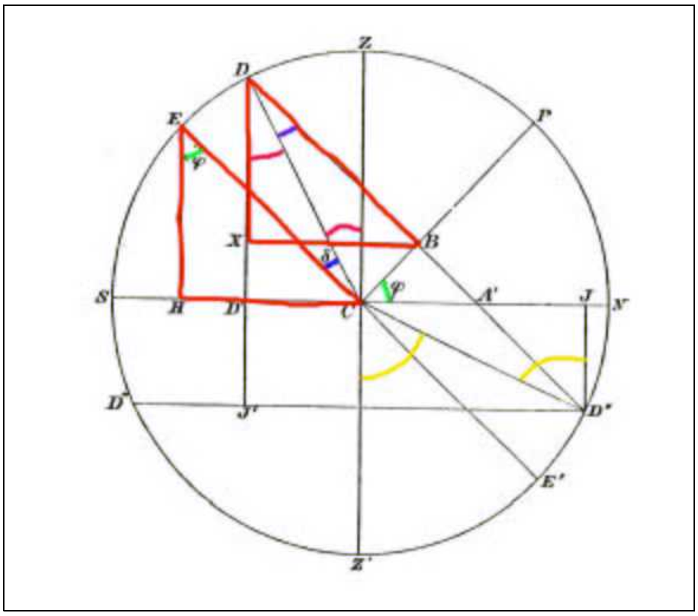


Figure 6-6 the projection of Figure 6-5 on the Meridian

And now for the proof of prosthaphaeresis:

From Figure 6-6 angle $\not\subset$ DCZ = $\varphi - \delta$, therefore DD' = cos ($\varphi - \delta$) and angle $\not\subset$ D''CZ' = $\varphi + \delta$, and therefore D'I = cos ($\varphi + \delta$).

If one draws D'' D''' parallel to NS, which the extension of DD' cuts at J', then DJ' = cos ($\varphi - \delta$) + cos ($\varphi + \delta$) and if BX is drawn parallel to NS , then DX = ½ [(cos ($\varphi - \delta$) + cos ($\varphi + \delta$)].

As DBX and ECH are similar triangles, it follows that

$$\frac{BD}{DX} = \frac{EC}{EH}$$

or in the case of the unit circle with EC = 1, the result is EH • BD = DX.

With BD = cos δ and EH = cos φ this equation becomes Formula 5:

Formula 5: cos φ • cos δ = ½ [(cos ($\varphi - \delta$) + cos ($\varphi + \delta$)]

Simpson [Simpson, page 120] presented a trigonometric method to convert the four prosthaphaeretic formulae into one another.

For example, to show {sin X • sin Y} as sum:

With $\varphi = 90° - X$ and $\delta = 90° - Y$, then from formula 5 we get

$$\cos(90° - X) \cdot \cos(90° - Y) = \tfrac{1}{2}[(\cos(90° - X - (90° - Y)) + \cos(90° - X + 90° - Y)]$$

$$\sin X \cdot \sin Y = \tfrac{1}{2}[(\cos(-X + Y) + \cos(180° - X - Y)]$$

$$\sin X \cdot \sin Y = \tfrac{1}{2}[(\cos(-(X - Y)) + \cos(180° - (X + Y))]$$

with cos (- Z) = cos Z and cos (180° - Z) = - cos Z it follows that:

Formula 6: sin X • sin Y = ½ [(cos (X - Y) - cos (X + Y)]

Another method makes use of the addition theorems [Enzykl]:

$$\sin(a + b) = \sin a \cos b + \sin b \cos a + \sin(a - b) = \sin a \cos b - \sin b \cos a$$

= [sin (a + b) + sin (a - b)] = 2 sin a cos b or

Formula 7: sin a • cos b = ½ [sin (a + b) + sin (a - b)] = A • B with A and B being the numeri for sin a and cos b.

With these easily understandable transformations of the various formulae, it might be hard to imagine that Johannes Werner did not know these methods.

To the extent that a distinction can be made between the discovery of the prosthaphaeretic formulae, and their application as prosthaphaeretic computing methods - that distinction can be designated (as does [Thoren]) as speculative and redundant. See also [Björnbo, page 157].

7 Applications

As a first application, an example of a multiplication of the numbers 0.6157 and 0.9397 is shown with one of the formulae shown in Figure 7 above, here restated:

$$\mathbf{A \bullet B = \sin a \bullet \cos b = \tfrac{1}{2}\, [\sin (a + b) + \sin (a - b)]}$$

with the factors A = sin a = 0.6157 and B = cos b = 0.9397 [Enzykl, page 70]. From the table in Fig 7-1 we read for factor sin a an angle of a = 38° in the green grads (degrees) column, and for cos b an angle of b = 20° from the red inverted grads (degrees) column (see red ellipses).

Grad	sin	tan	
0	0,0000	0,0000	90
1	0175	0175	89
2	0349	0349	88
3	0523	0524	87
4	0698	0699	86
5	0872	0875	85
6	1045	1051	84
7	1219	1228	83
8	1392	1405	82
9	1564	1584	81
10	0,1736	0,1763	80
11	1908	1944	79
12	2079	2126	78
13	2250	2309	77
14	2419	2493	76
15	2588	2679	75
16	2756	2867	74
17	2924	3057	73
18	3090	3249	72
19	3256	3443	71
20	0,3420	0,3640	70
21	3584	3839	69
22	3746	4040	68
23	3907	4245	67
24	4067	4452	66
25	4226	4663	65
26	4384	4877	64
27	4540	5095	63
28	4695	5317	62
29	4848	5543	61
30	0,5000	0,5774	60
31	5150	6009	59
32	5299	6249	58
33	5446	6494	57
34	5592	6745	56
35	5736	7002	55
36	5878	7265	54
37	6018	7536	53
38	6157	7813	52
39	6293	8098	51
40	0,6428	0,8391	50
41	6561	8693	49
42	6691	9004	48
43	6820	9325	47
44	6947	9657	46
45	7071	1,0000	45
	cos	cot	Grad

Grad	sin	tan	
45	0,7071	1,000	45
46	7193	036	44
47	7314	072	43
48	7431	111	42
49	7547	150	41
50	0,7660	1,192	40
51	7771	235	39
52	7880	280	38
53	7986	327	37
54	8090	376	36
55	8192	428	35
56	8290	483	34
57	8387	540	33
58	8480	600	32
59	8572	664	31
60	0,8660	1,732	30
61	8746	804	29
62	8829	881	28
63	8910	963	27
64	8988	2,050	26
65	9063	145	25
66	9135	246	24
67	9205	356	23
68	9272	475	22
69	9336	605	21
70	0,9397	2,747	20
71	9455	904	19
72	9511	3,078	18
73	9563	271	17
74	9613	487	16
75	9659	732	15
76	9703	4,011	14
77	9744	331	13
78	9781	705	12
79	9816	5,145	11
80	0,9848	5,671	10
81	9877	6,314	9
82	9903	7,115	8
83	9925	8,144	7
84	9945	9,514	6
85	9962	11,43	5
86	9976	14,30	4
87	9986	19,08	3
88	9994	28,64	2
89	9998	57,29	1
90	1,0000	∞	0
	cos	cot	Grad

That is, for the two figures in the blue boxes we see the results for a + b = 58° and for a - b = 18°.

With sin 58° = 0.8480 and sin 18° = 0.3090 their sum is 1.1570. Halving this gives 0.5786, which is the correct product of 0.6157 • 0.9397 to 4 places (to be more exact: 0.57857329).

It is easy to recognise that using a sine table with higher precision will result in a higher accuracy.

Using Regiomontanus' seven-figure tables from his sine tables of 1541 we find sin 18° = 0.3090170 and sin 58° = 0.8480481, giving a sum of 1.1570651; halving that value gives a result of 0.578532550 – a value now quite close to the real product.

Figure 7-1 four-figure table from [Enzykl, page 805]

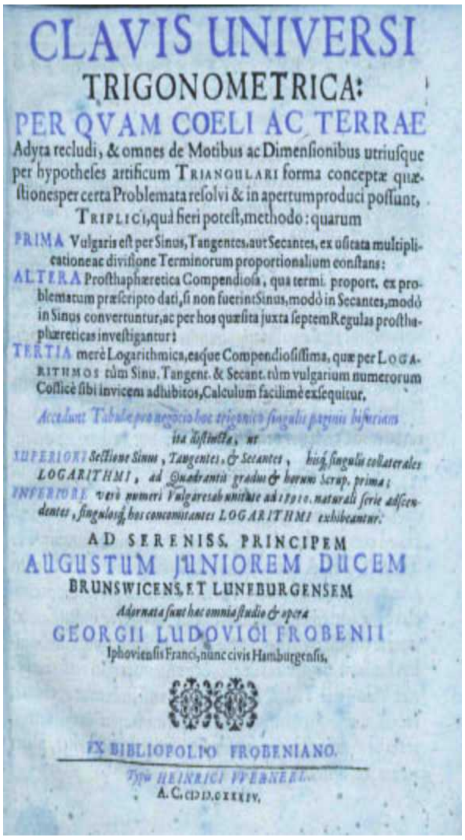
It is also possible to do division is this manner. Replacing cos b by 1/sec b and thereby getting another angle for b, it can be further calculated with the right side of the formula 7.
An example for this is in [Sher].

As the connoisseur can imagine, it is also possible to reverse this process to, for example, calculate an addition by using a multiplication. This method is very geeky, but might be of theoretical interest to a slide rule user!

George Ludwig FROBENIUS (25.8.1566 in Iphofen - 21.7.1645 in Hamburg) was a Polyhistor (Universal scholar), a mathematician, a bookseller and a Hamburg publisher.

He lived in a time of change in computing methods as used in astronomical applications.

These circumstances were shown in his ***Clavis Universi Trigonometrica*** [Frobenius] in which arithmetical examples of the known methods of calculation were presented.


CLAVIS UNIVERSI
TRIGONOMETRICA:
PER QVAM COELI AC TERRAE
Adyta recludi, & omnes de Motibus ac Dimensionibus utriusque per hypotheses artificum TRIANGULARI forma conceptæ quæstiones per certa Problemata resolvi & in apertum produci possunt, TRIPLICI, quâ fieri potest, methodo: quarum

PRIMA Vulgaris est per Sinus, Tangentes, aut Secantes, ex usitata multiplicatione ac divisione Terminorum proportionalium constans:

ALTERA Prosthaphæretica Compendiosa, qua termi. proport. ex problematum præscripto dati, si non fuerint Sinus, modò in Secantes, modò in Sinus convertuntur, ac per hos quæsita juxta septem Regulas prosthaphæreticas investigantur:

TERTIA merè Logarithmica, eaque Compendiosissima, quæ per LOGARITHMOS tùm Sinu. Tangent. & Secant. tùm vulgarium numerorum Cosllicè sibi invicem adhibitos, Calculum facilimè exsequitur.

Accedunt Tabulæ pro negocio hoc trigonico singulis paginis bifariam ita distinctæ, ut

SUPERIORI *Sectione Sinus, Tangentes, & Secantes, hisq; singulis collaterales LOGARITHMI, ad Quadrantis gradus & horum Scrup. prima;*

INFERIORE *verò numeri Vulgares ab unitate ad* [illegible] *naturali serie adscendentes, singulosq; hos concomitantes LOGARITHMI exhibeantur.*

AD SERENISS. PRINCIPEM
AUGUSTUM JUNIOREM DUCEM
BRUNSWICENS. ET LUNEBURGENSEM
Adornata sunt hæc omnia studio & opera
GEORGII LUDOVICI FROBENII
Iphoviensis Franci, nunc civis Hamburgensis.

EX BIBLIOPOLIO FROBENIANO.
Typis HEINRICI WERNERI.
A.C. CIƆ IƆ.CXXXIV.

Figure 7-2 2nd title page Frobenius

Frobenius used three methods, which he named

- "Prima (1^{st})" or "Vulgaris",
- "Altera (2^{nd})" or "Prosthaphaeretice"
- "Tertia (3^{rd})" or "Logarithmice".

In the following examples, the three different methods are demonstrated and described to demonstrate the computation of an elevation, which results from the cutting across two diameters (great circles around a sphere).The two diameters are taken from astronomy (spherical trigonometry) and represent the equator and the diameter of the Earth in line with the ecliptic.

TRIGONOMETRIA SPHÆRICORUM. 43

TRIGONOMETRIÆ PRACTICÆ,

SECTIO ALTERA.

PROBLEMATA XXVI. TRIANGULORUM

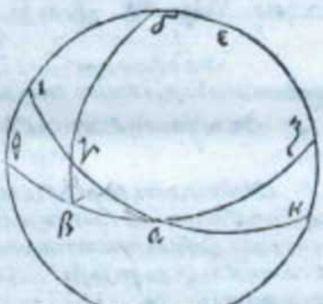
Sphæricorum analysin spectantia continens.

PRIMUM PROBLEMA.

TRIANGULI Sphærici rectanguli datâ, præter rectum, Basi cum Angulo ad Basin obliquo, CRUS dato angulo subtensum invenire, Modis sex, & horum singulos trifariam absolvere, *Vulgariter, Prosthaphæreticè & Logarithmicè.*

PRIMUS MODUS.

Ut Sinus totus est ad Sinum Basis : Ita Sinus anguli dati ad Sinum Cruris oppositi. *16.p.4.Regiom.3.e.14.F.1.mf.12.e.4.L.2.probl.41.p.Clav.de triang.Sphær.*



EXEMPLUM: *Dantur in præsenti Triang. γβα Sphærico rectang. præter Rectum ad β, basis αγ arcus Eclipticæ 30.gr.8.pr.55.sec.& angulus γαβ obliquitatis Eclipticæ hoc ævo 23.gr.31.pr.30.sec: Quæritur Crus βγ.*

VULGARITER.

T.P. γβα 90.gr. — γα 30.gr.8.pr.55.sec. — γαβ 23.gr.31½.pr.
S.t.10,000,000 — Si.5,022,446 — Si.3,991,492.
Quartus 2,004,705. Sinus: Ejus arcus 11.gr.33.pr.52.sec. Crus βγ, Declinat. dati puncti Eclip.

PROSTHAPHÆRETICE. Consulatur PRIMA Regula Prosthaphæretica.

Arcus major 30.gr.8.pr.55.sec. Compl.59.gr.51.pr.5.sec.
Arcus minor. 23. 31. 30. - - - 23. 31. 30.

Aggreg. 83. 22. 35. Sin. 9,933,253.
Differ. 36. 19. 35. Sin. 5,923,839 subt.

Differ. 4,009,414.
Semis 2,004,707 Sinus, hujus arcus est 11.gr.33.pr.52.sec. Crus βγ.

F 2 Loga-

44 TRIGONOMETRIA SPHÆRICORUM.

LOGARITHMICE: Ut Logarithmus Sinus anguli recti est ad Logarithmum Sinus Basis : Ita Logarithmus Sinus anguli, ad Logarithmum Sinus Cruris quæsiti.

EXEM. *T.P. γβα 90.gr. — γα 30.gr.8.pr.55. — γαβ 23.gr.31.pr.30.sec.*
Log.Si.10,000,000 — Log.Si.9,700,915 — Log.Si.9,601,136.
9,601,136 +
19,302,051
10,000,000 —
Log.Sinus 9,302,051 : arcus 11.gr.33.pr.52.se. Crus βγ.

Figure 7-3 Example pages from Frobenius for the computation of a side in a spherical triangle

In the spherical triangle **αβγ**, the angle **a** opposite the side **βγ** is to be computed.

Two angles and the side opposite the other angle (**ß**) are given.

The characteristic (and simplification) is that this concerns a right-angled spherical triangle; the right angle (90°) is at the angle **ß**.

For the computation of the side **βγ**, the sine rule is applicable; within a right-angled spherical triangle this is expressed as the

Sine rule: sin a = sin α • sinγα / sin ß

In Frobenius´example, the angle **α** has a value of 23 degrees, 31 minutes and 30 seconds - the angle of the ecliptic. The side **γα** has a value of 30 degrees, 8 minutes and 55 Seconds. Because sin ß = sin 90° = 1, the formula simplifies to:

sin a = sin α • sin γα

Following the first "Vulgariter" method, the results of the calculation process are presented in table 7-1:

Table 7-1 Calculation method using multiplication of sines [4]

1. Vulgariter	Angle/Side	Trig Funct	Degree	Pr	Sec	
	α		23	31	30	
	γα = b		30	8	55	
	β		90			
	Si α	3.991.492				
		times				
	Si γα	5.022.446				
		divided by				
	S.t. (sinus totus = β)	10.000.000				
		equals				
	Quartus = si α x si γα	2.004.705				
Solution:	Sinus: arcus ejus		11	33	52	Crus βγ

According to this method the seven-place sines of the angles were determined and **multiplied** with one another.

The solution for the side **ßγ = a** gives: 11 degrees, 33 minutes 52 seconds.

The 2nd method is "Prosthaphaeretic", which works according to the following formula:

$$\sin \gamma a \bullet \sin \alpha = \frac{1}{2}\{\sin((90° - \gamma a) + \alpha) - \sin((90° - \alpha) - \gamma a)\}$$

Table 7-2 Calculation method using Prosthaphaeresis

2. Prosthaphäretice	Angle/Side	Degree	Pr	Sec			Degree	Pr	Sec		Sinus	Arcus		
γα	Arcus Major	30	8	55		Compl γα	59	51	5			Grad	Pr	Sec
α	Arcus Minor	23	31	30		α	23	31	30					
					plus	Aggreg.	83	22	35		9,933,253			
										minus				
					minus	Differ.	36	19	35		5,923,843			
										Differ.	4,009,410			
										divided by 2				
Solution:						Crus βγ				Semis	2,004,705	11	33	52

[4] It should be mentioned here that the trigonometric calculations from Frobenius are based on sinus totus = sin 90° = 10^7

In table 7-2 the calculation method is shown, in which the result is calculated using the prosthaphaeretic formula. To remind the reader: the product to be computed (sin **γα** times sin **α**) can be computed by the **Addition and Subtraction** of Sines. Only at the conclusion there is an additional, simple, division (Semis) by 2. We see here a somewhat elaborate calculation process like the above Vulgariter method; however simpler calculation steps are used.

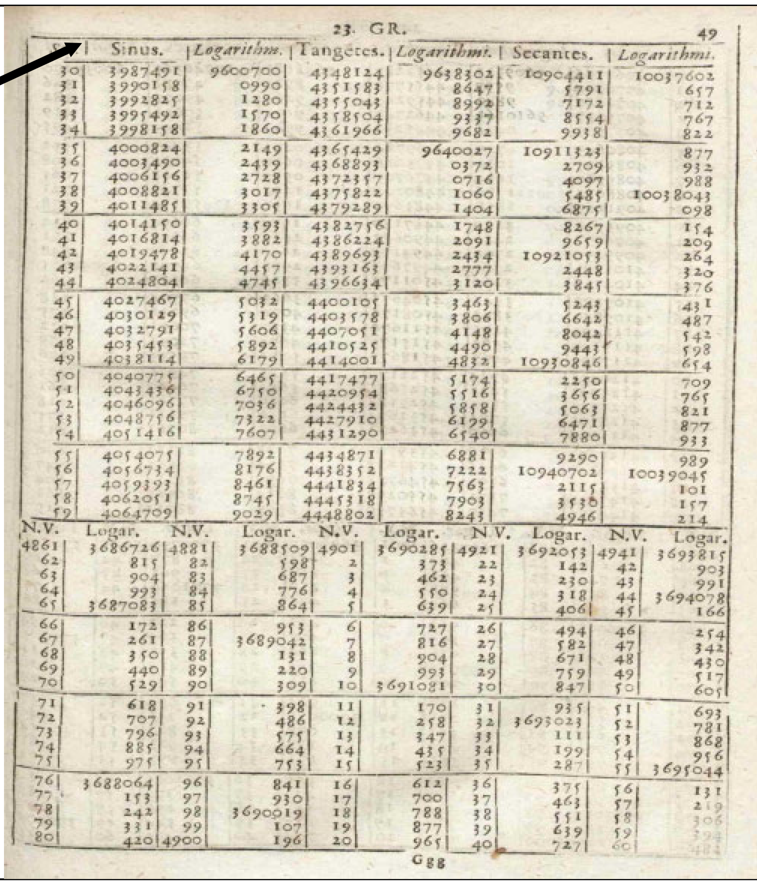
The simplest and fastest way for the computation of the height is the logarithmic method shown in table 7-3. In addition it was necessary, to look up the logarithms associated with the sines and **add and/or subtract** these. The use of suitable tables was trusted by astronomers of that time, because there already existed appropriate tables for the trigonometric functions and for their logarithms.

Table 7-3 Calculation method using Logarithms

3. Logarithmice	Angle/Side	Degree	Pr	Sec	SIN (:10^7)	log sin ∢		log sin (+10)	Degree	Pr	Sec
	α	23	31	30	0,3991492	-0,39886476		9,60113524			
	γα	30	8	55	0,5022445	-0,29908475	plus	9,70091525			
								19,30205049			
	β	90			1,0000000		minus	10,00000000			
Solution:							Crus βγ	9,30205049	**11**	**33**	**52**

Frobenius had included a detailed table (see Figure 7-4) in his comprehensive *Clavis Universi Trigonometrica* (323 pages text book plus 184 pages of tables); in this table, sines as well as tangents and secants, and their logarithms, for each minute of angle are exactly set down. Additionally, the Briggsian logarithms of the numbers are tabulated.

Sinus. = SIN = 10^7 sin



23. GR. 49

	Sinus.	Logarithm.	Tangetes.	Logarithmi.	Secantes.	Logarithmi.
30	3987491	9600700	4348124	9638302	10904411	10037602
31	3990158	0990	4351583	8647	5791	657
32	3992825	1280	4355043	8992	7172	712
33	3995492	1570	4358504	9337	8554	767
34	3998158	1860	4361966	9682	9938	822
35	4000824	2149	4365429	9640027	10911323	877
36	4003490	2439	4368893	0372	2709	932
37	4006156	2728	4372357	0716	4097	988
38	4008821	3017	4375822	1060	5485	10038043
39	4011485	3305	4379289	1404	6875	098
40	4014150	3593	4382756	1748	8267	154
41	4016814	3882	4386224	2091	9659	209
42	4019478	4170	4389693	2434	10921053	264
43	4022141	4457	4393163	2777	2448	320
44	4024804	4745	4396634	3120	3845	376
45	4027467	5032	4400105	3463	5243	431
46	4030129	5319	4403578	3806	6642	487
47	4032791	5606	4407051	4148	8042	542
48	4035453	5892	4410525	4490	9443	598
49	4038114	6179	4414001	4832	10930846	654
50	4040775	6465	4417477	5174	2250	709
51	4043436	6750	4420954	5516	3656	765
52	4046096	7036	4424432	5858	5063	821
53	4048756	7322	4427910	6199	6471	877
54	4051416	7607	4431290	6540	7880	933
55	4054075	7892	4434871	6881	9290	989
56	4056734	8176	4438352	7222	10940702	10039045
57	4059393	8461	4441834	7563	2115	101
58	4062051	8745	4445318	7903	3530	157
59	4064709	9029	4448802	8243	4946	214

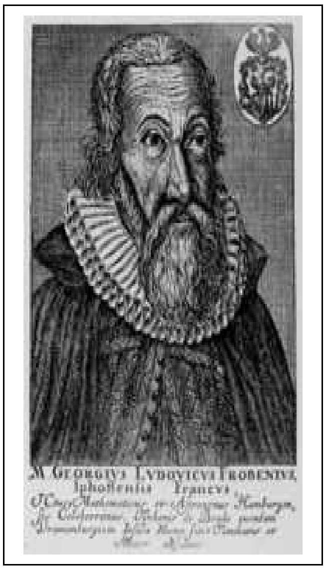
N.V.	Logar.	N.V.	Logar.	N.V.	Logar.	N.V.	Logar.	N.V.	Logar.
4861	3686726	4881	3688509	4901	3690285	4921	3692053	4941	3693815
62	815	82	598	2	373	22	142	42	903
63	904	83	687	3	462	23	230	43	991
64	993	84	776	4	550	24	318	44	3694078
65	3687083	85	864	5	639	25	406	45	166
66	172	86	953	6	727	26	494	46	254
67	261	87	3689042	7	816	27	582	47	342
68	350	88	131	8	904	28	671	48	430
69	440	89	220	9	993	29	759	49	517
70	529	90	309	10	3691081	30	847	50	605
71	618	91	398	11	170	31	935	51	693
72	707	92	486	12	258	32	3693023	52	781
73	796	93	575	13	347	33	111	53	868
74	885	94	664	14	435	34	199	54	956
75	975	95	753	15	523	35	287	55	3695044
76	3688064	96	841	16	612	36	375	56	131
77	153	97	930	17	700	37	463	57	219
78	242	98	3690019	18	788	38	551	58	306
79	331	99	107	19	877	39	639	59	394
80	420	4900	196	20	965	40	727	60	481

Ggg

Logarithm. = LOG (SIN) = $10^7 + 10^7$ log sin = 10^7 (1 + log (sin))

Figure 7-4: Excerpt from a table by [Frobenius] showing 23 degrees and 31 minutes.

Frobenius leaves it to the reader as to which computation method might be used. However, given that very extensive and very detailed six and/or seven digit tables for both trigonometric functions and for logarithms were available, there was a preference to use logarithms as the latter were better known.



George Ludwig Frobenius (1566 - 1645)

Link as at 25th February 2012: http://de.wikipedia.org/wiki/Georg_Ludwig_Frobenius

Another application was only recently published – “the construction of a ‘prosthaphaeretic slide rule’” [Sher]. The principle is shown in the following series of figures.

Figure 7-5: the principle of the "prosthaphaeretic" slide rule

Of the many ways to prove (3), the following is the most useful in these circumstances because it suggests clearly how to make the prosthaphaeretic slide rule. It is sufficient for our purposes to consider only the case where A and B are acute angles, $A > B$, and $A + B < 90$ degrees. Figure 1, which follows, is of the unit circle in the first quadrant.

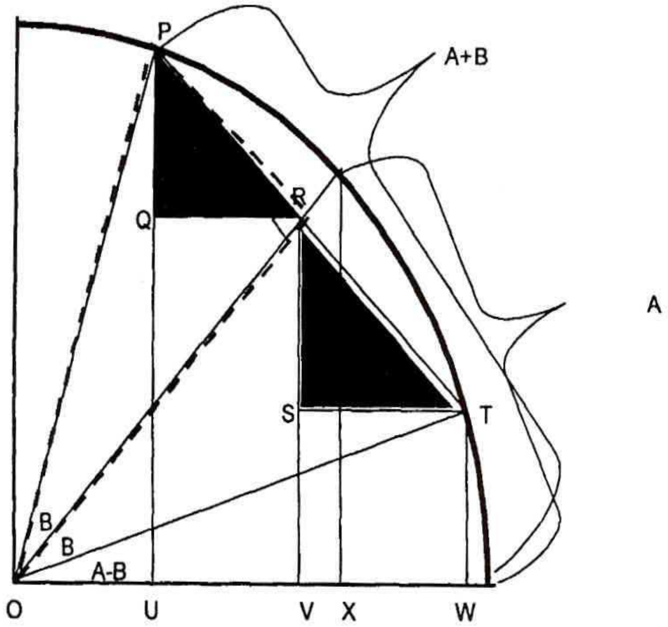


Figure 1

In Figure 1, $\overline{OU} = \cos(A+B), \overline{OW} = \cos(A-B)$ and $\overline{OR} = \cos B$ (use right triangle trigonometry on triangle ORP). Since triangles PQR and RST are congruent, it follows that $\overline{QR} = \overline{ST}$ and, thus, $\overline{UV} = \overline{VW}$. Since V is the midpoint of $\overline{UW}$, it follows that

Side calculation: OV = OU + UV (1) and OV = OW - WV = OW - UV,
therefore UV = WV.
With UV = OW - OV in (1) the result is: OV = OU + OW - OV - > 2 OV = OU + OW
and thus OV = [OU + OW] /2

Figure 7-6: continuation of the proof and construction guidance for the prosthaphaeretic slide rule

$$\overline{OV} = \frac{\cos(A+B)+\cos(A-B)}{2}$$

Using right triangle trigonometry on triangle OVR, we get $\overline{OV} = \overline{OR}\cos A = \cos B \cos A$ (since $\overline{OR} = \cos B$ from right triangle trigonometry on ORP). Thus,

$$\cos A \cos B = \overline{OV} = \frac{\cos(A+B)+\cos(A-B)}{2}.$$

Figure 2 shows that similar triangles can also justify the design of our device. The proof uses the fact that the ratio of b to 1 is the same as the ratio of ab to a.

CONSTRUCTING THE PROSTHAPHAERETIC SLIDE RULE

Figure 1 now tells us how to multiply two numbers that lie between zero and one. Notice that $(\overline{OX})(\overline{OR}) = \overline{OV}$, since $\overline{OX} = \cos A$, $\overline{OR} = \cos B$ and $\overline{OV} = \cos A \cos B$. If we let $\overline{OX} = a$, $\overline{OR} = b$, and remove the now unnecesssary parts of Figure 1, we get Figure 2.

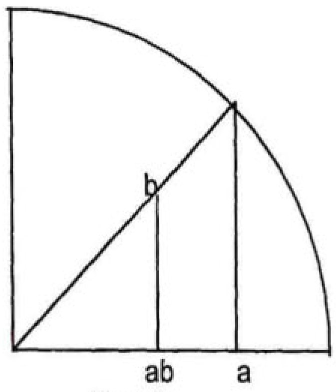


Figure 2

This figure shows how to construct the device:

1) Draw the first quadrant of the unit circle on a square surface. On the horizontal axis place a unit scale where 0 is at the origin and 1 is the radius of the unit circle. Subdivide the scale as finely as possible (tenths, hundredths, etc.)

2) Attach a rotor at the origin which has the same scale on it as the horizontal axis. The zero on this scale must be at the origin.

3) Attach to the square surface a slider that maintains a line perpendicular to the horizontal axis and can be moved left and right (much like the slider on a slide rule) always maintaining its perpendicularity.

Figure 7-7 application of the prosthaphaeretic slide rule

The result is as follows:

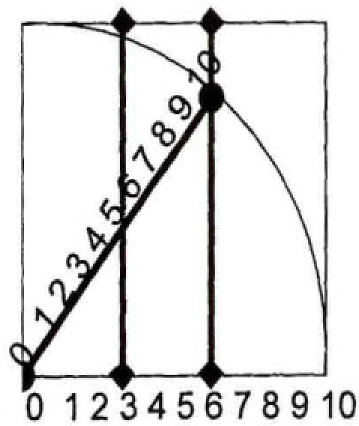


Figure 3: Scales in tenths

We can now see that multiplying two numbers between 0 and 1 is easy. Let's do 0.6×0.5 as an example.

1) Position the slider so that it passes through 0.6 on the horizontal axis.
2) Position the rotor so that its 1 coincides with the slider.
3) Without disturbing the rotor's position, move the slider until it passes through 0.5 on the rotor's scale.
4) The product (0.3) is at the intersection of the slider and the horizontal axis.

As Example 1 shows, any multiplication can be reduced to multiplying a pair of numbers between 0 and 1 and then adjusting the decimal point.

The device also does divisions. As an example of how to do $\frac{y}{x}$ when $0 < y < x < 1$, let's do $0.3 \div 0.5$.

1) Position the slider so that its vertical line passes through 0.3 on the horizontal axis.
2) Position the rotor so that 0.5 on its scale coincides with the slider.
3) Without disturbing the rotor's position, move the slider until it passes through 1 on the rotor's scale.
4) The answer (0.6) is at the intersection of the slider and the horizontal axis.

Figure 3 contained in Figure 7-7 points to the application of the intercept theorems, convenient for doing calculation by proportions, and one has thereby naturally removed oneself far from actual Prosthaphaeresis. Therefore, the above written steps represent the geometrical transition of the prosthaphaeretic formulae to intercept theorems.

Thus it is not surprising, that some computing devices, well-known for some time, use this principle of the so-called “prosthaphaeretic slide rule”, without having anything to do with either Prosthaphaeresis or with logarithms (as used on slide rules).

For example, in 1914, M. Cashmore in England already had a patent for a technically quite beautiful looking proportional computer, which was patented under GB 1914-13073 and is described in details in the online computer encyclopaedia [Relex].

The Flash Animation, which can be seen there, shows the computing principle of this proportional computer impressively.

During the operation of this, there is no computational meaning assigned to the hypotenuse. However, Sher and Nataro [Sher], using their method, read the output from the hypotenuse, because they extract the cosines of the angles A and B.

Cashmore receives his multiplication result on the movable opposite side of the triangle after only one rotation. Such an application of intercept theorem thus brings a simplification.

However, according to a comment by a well-known lady Mathematician, "the Prosthaphaeretic Slide rule has about as much to do with Prosthaphaeresis as a set-square has with a Toblerone box."

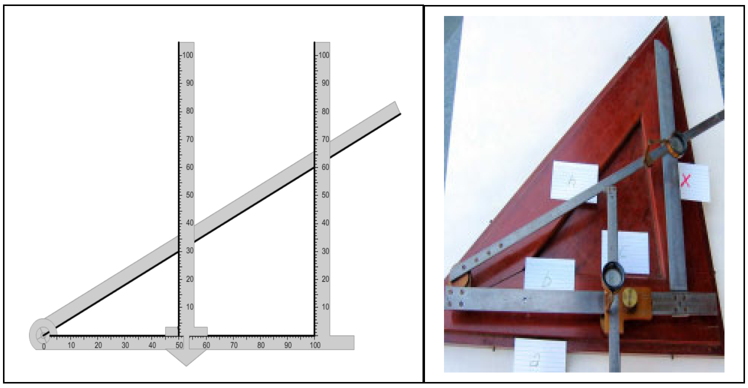


Figure 7-8 computing principle of the Cashmore of proportional computer [from Relex]

According the to intercept theorems, the following is valid: $a/b = X/c$

With **a** having unit value (1 or 10 or 100), it follows that: $c/b = X/1$

Or $c = X \cdot b$

It is necessary to consider the decimal places of the results. So, for example, in Figure 7-8:

$X = 60$, $b = 50$, results in $c = 30 \cdot 100 = 3000$.

Since the 17th Century, proportional calculating instruments, such as the sector and the proportional divider/circle developed by Jost Bürgi, make wide-spread appearances.

Nicholas Rose [Rose] has described further applications of Prosthaphaeresis. In one such, he applies Prosthaphaeresis to music, to explain the theory of vibrations and beats; in another, he explains why it is not possible to receive high-fidelity reproduction using a signal from a medium-wave transmitter.

The prosthaphaeretic method was used for about hundred years with great success, because it represented a considerable aid to computation. Moreover, to judge from the works of Longomontan and Frobenius [Frobenius], several mathematicians continued to use their trusted Prosthaphaeresis even after the publication of the logarithms.

Others however concerned themselves with the logarithms and valued their use very highly: The astronomer and mathematician Marquis Pierre-Simon de Laplace (1749 – 1827) claims that

"The invention of the logarithms shortens calculations which might have lasted for months, to a few days, doubling thereby the life of the (human) computers."

Moreover, the application of prosthaphaeresis had been able to contribute for some decades and it may have been the trigger for the emergence of logarithms, because John Napier (1550-1617) and Jost Bürgi (1552 - 1632), the inventors of logarithms, were users of prosthaphaeresis. Bürgi used prosthaphaeresis for his computations concerning his observations of Mars around 1590 [Faustmann]. According to Volker Bialas [Bialas] Johannes Kepler (1571-1630) also used prosthaphaeresis for his calculations for his "Epitome Astronomiae Copernicanae" (Outline of Copernican Astronomy - 1618/1621).

8 Time tables and Summary

Prosthaphaeresis text history time table (after Björnbo 1907)

Year	Event	Björnbo B Page #; **v. Braunmühl vB Page#**
6.06.1436	**Johannes Mueller is born in Königsberg near Hassfurt (Johann de Monte Regio; Regiomontanus), was for 15 years a student under Georg Peurbach.**	**vB118**
1446	Sebald Schreyer, a patron of Werner, is born	B153
1468	Cardinal Matthäus Lang, Bishop of Gurk, is born, thereafter he became Archbishop of Salzburg; Sponsor of Werner (died 1540).	B153
14.02.1468	**Johannes Werner born in Nuremberg.**	**B150**
1470	Willibald Pierckheimer born in Nuremberg born; Patron of Werner; a member of a group of Nuremberg humanists.	B153
6.07.1476	**Johannes Regiomontanus dies in Rome: "his work was so great that he towers above his contemporaries and stands as a milestone marking the beginning of the new age"**	**vB119**
1477	Johannes Schöner born.	B153
1489	Georg Hartmann born	B158
1493 - 1497/8	Werner goes to Italy - interestingly, the catalogue entry for the Cod Reg. in 1259 carries the year 1495; however, this cannot be correct in that quotations from later dates are included within it. (It is possible, however, that these were inserted later).	B150
1497/8	Johannes Werner returns to Nuremberg and becomes a parish priest there.	B150
1504	The year of Bernhard Walther's death, who administered the legacy of Regiomontanus ("kept fearfully locked away"). Early in this year, Werner gains access to Regiomontanus' work, and expresses his annoyance that he needed so much time in his hometown of Nuremburg (in which Walther lived and Regiomonantanus worked) before he could gain this access.	B173; vB132
1505	Starting from this year, Werner's work was drawn up and his friend and patron Willibald Pirckheimer acquired the triangle books of Regiomontanus (and probably passed them on to Werner). Werner does not give any reference to Regiomontanus' work in his own work. He mentions however Euclid, Menelaus, Theodosios, Geber and Georg von Peuerbach and it seems likely, given that their work formed the basis of what Regiomontanus had written, that this is probably why Werner did not give any references to Regiomontanus' work. **However, we know the carelessness of those times in the taking of sufficient care of intellectual property rights, to assume with good reason that Werner might have absorbed some of Regiomontan's work into his own trigonometry books, using a handed-down original. As an example of this, the transformation of Regiomontan's Cosine Rule by the invention of the so-called prosthaphaeretic method, albeit that the former was from a much earlier time, became Werner's intellectual property. "**	B172, B173; **vB135**
1505-1513	Werner writes his 5 books concerning spherical triangles (*Liber de triangulis sphaericis*) - ***Spherical trigonometry in 5 books under the title "De triangulis per maximorum circulorum segmenta constructis libri V ", for which he unfortunately found no publisher.***	B157; **vB133**
1514	Werner has now been a parish priest for 16 years; this is the year of appearance of his first compilation.	B150
1514	George Joachim Rheticus born.	B158
1516	Konrad Gessner born.	B158
1446	Sebald Schreyer, a patron of Werner, died.	B153
1522	Werner's second compilation appears, including his work after 1513; Title: In hoc opere haec continentur. Libellus Ioannis Verneri Nurembergen. Super vigintiduobus elementis conicis. Ejusdem: Commentarius seu paraphrastica enarratio inundecim modos conficiendi eius problematis quod cubi duplicatio dicitur etc etc.	B153; See Title page from the Internet
1522	Johannes Stabius, Viennese mathematician and Imperial Histiograph; Patron of Werner, with whose help the 1514 compilation appeared, died.	B153
1525	Kaspar Wolf born.	B158
1525-1522	Werner is now a priest at the cemetery of St. Johannes	B151
1522	**Johannes Werner dies in Nuremberg, without one of his works ever having appeared in print.**	**B150, B157**
1530	Willibald Pierckheimer dies in Nuremberg.	B153
1533	There appears a new publication by Peter Apian of his *Introductio Geographica*	B158

	together with a letter from the year 1532, which he received from Johann Wilhelm of Loubemberg, with the request that his own book on the subject of Meteoroscopy be permitted to be published, which happens. It is supposed that Apian's Work is a compilation from Werner's first two books.	
1537	Johann Richter (also known as Praetorius) is born.	B170
1542	During a visit to Nuremberg, Hartmann hands over Werner's written legacy to Rheticus (according to Doppelmayr 1730); this legacy is described as disordered sheets of paper, like the rubble of a shipwreck.	B158, B160; **vB133**
1546	Publication of Werner's *Metereologi Canones* by Schöner	B153
1546	Tycho Brahe born	B165
1547	Johannes Schöner dies - **was to have published in 1557 in Neuburg a further Regiomontanus Work "Fundamentum Operationem…"..**	B153; **vB123**
1554	Jakob Christmann born	B165
1555	Konrad Gessner, Bibliographer, says that the Nuremberg Mathematician and Mechanic Georg Hartmann saves Werner's six books concerning Meteoroscopy from destruction.	B158
1555?	Paul Wittich, future pupil of Tycho Brahe, born.	B167
1557 Note	In Kraków a book appears, containing on its title page, the titles of the two Werner works, and giving as its publisher Rheticus - "Nunc Primum Editi" thus 1557 not published before 1557 (see the title page Figure 5.3 in the main text) - found by Eneström in 1902 and Zebrawski in 1873. Rheticus afforded Werner's work considerable respect and also indicates him as the source, which others (for example, Peter Apian) did not do. Moreover, it is assumed that Rheticus admitted that Werner's arrangement of the different triangle forms was the inspiration for his Opus Palatinum (1596).	B159 B162
1564	George Hartmann dies	B158
1565	Konrad Gessner dies	B158
1566; 1568-1569; 1575	Brahe is in Wittenberg and could have seen Werner´s work about triangles here. The formula used by Brahe and Wittich is identical with that which Reimers used; however it is not the same as Werner's, as the latter started with *Sinus Versus*.	B169
1568	Rheticus has completed half of his giant work Opus Palatinum.	B164
1569	Praetorius visits Rheticus in Kraków and sees the tables which appeared later in Opus Palatinum; at the same time, he probably also had a look at Werner's 4th book about Spherical Geometry.	B170
1571-1576	Praetorius is Professor of higher mathematics in Wittenberg	B170
1574 and 1583	Kaspar Wolf indicates in Gessner's new bibliography that, if he is not wrong, George Joachim Rheticus has edited the six books concerning *Meteoroscopy* and the four *de triangulis* books.	B158
1576	George Joachim Rheticus dies in Kaschau, Hungary - up to then the printed manuscript of Werner's works had been in his possession.	B158, B165
1576	L. Valentin Otho (born around 1550 in Magdeburg), the pupil of Rheticus, receives Rheticus' legacy (with Werner manuscripts) after his death.	B165
1580	"Re-invention" (according to v. Braunmühl) of Prosthaphaeresis by Tycho Brahe and Paul Wittich in Uranienborg on Hven.	B167, B168; **vB193**
1584	**In this year a certain Paul Wittich (1555? - 1587) comes to the court of Count Wilhelm IV (1532 - 1592) from Wroclaw; he had stayed with Tycho at Uranienburg from 1580 to 1581 and thereafter remains for a longer time in Kassel.** **As reported by both Bürgi and Raymarus Ursus, he (Wittich) has the first case of the so-called Prosthaphaeretic Method, which as we have learned from P 135, was one of Werner's inventions. The astronomers at Kassel showed the Prosthaphaeretic Method to be a calculation method which was used for a long time at Uranienburg for astronomical calculations, albeit without proof.** **Ursus goes on to explain that Bürgi found such a fruitful proof for this, that using it, the other prosthaphaeretic case and his (Ursus') proof, indeed the solution of all triangles by this method can be deduced by means of sines, tangents and secants.**	**vB194; vB195**
1587	Paul Wittich dies	B167
1588	Nicolai Reimers alias Nikolaus Raymarus Ursus publishes his *Fundamentum Astronomicum* in Strasbourg; in it Werner's Prosthaphaeresis work **(both prosthaphaeretic rules including their derivative in two diagrams)** is published for the first time.	B168, B171; **vB195**
1589/1590	Brahe complains in letters to Hayck that Nicolai Reimers has stolen his Prosthaphaeretic Formulae…	B168, B169
1590	… however without the appropriate mathematical proofs. These formulae are received by Reimers from Jobst Bürgi in Kassel, to where they are allegedly brought by Paul Wittich.	B169

1599	Nicolai Reimers dies.	B169
1599	**Johann Praetorius announces in his Cod. lat. monac. 24101 that Werner is the inventor of the prosthaphaeretic equation, which some certain pupil brought into a somewhat simplified form.**	**vB137 Footnote 2**
1601	Kaspar Wolf dies	B158
1601	Tycho Brahe dies	B165
1611	In Jakob Christmann's *Theoria Lunae* the two works of Werner are quoted and a form of the Prosthaphaeresis used, to find the angle B of the obtuse spherical triangle when sides a, b, c are known- “. **.and that he further explained the method described therein using three diagrams. “**	B165; **vB136**
1613	Jakob Christmann, professor, dies - (not in 1630 as indicated elsewhere).	B165
1616	Johann Richter alias Praetorius dies	B170
Between 1654 and 1689	Queen Christina receives Werner's manuscripts, which up till then had been in the possession of Jakob Christmann. After her death in 1689 the manuscript (Cod. Regin. 1259) lay largely un-noticed in the Vatican.	B171
1622	Longomontan has attributed the invention of the Prosthaphaeresis in his *Astronomica Danica* of 1622 (page 10) to Paul Wittich, and was thereby the reason that all historians who came later had the same opinion. Rheticus had possession of Werner's manuscripts up to his death (1576) in Kraków and possibly used them as support for his *“Opus Palatinum”*. The manuscript did not contain mathematical diagrams, which suggests that the document was provided as a printer's proof. **As to the role of Jost Bürgi with his calculations using the prosthaphaeretic method and their proofs, there is probably more to be said!**	vB136 footnote 2 vB145 **vB194ff**
1622	**The method of the Prosthaphaeresis was so popular that it remained in use for a long time after the invention of the logarithms (1614). Such usage was recommended by the Dane Christian Severin Longomontan (1562 - 1647), who was a long-term assistant to Tycho and became, after the latter's death, a most enthusiastic Professor of Mathematics in Copenhagen.** **In his “Astronomica Danica” of 1622, already mentioned above, he even gives the advantage which he sees over logarithms, (which were well-known to him). “because by using Prosthaphaeresis the Rule of Three can be carried out handling only the given data", while “the procedure with logarithms is too far removed from current proofs, which is more important perhaps to beginners ".**	**vB202**
1634	**A proof for this, that astronomers could only with difficulty be separated from their beloved Prosthaphaeretic Method, is that Georg Ludwig Frobenius of Hamburg (1566 - 1645) published in 1634 his “Clavis Universi Trigonometrica " - exact title:** "Clavis universi trigonometrica : per quam coeli ac terrae adyta recludi, et omnes de motibus ac dimensionibus utriusque per hypotheses artificum triangulari forma conceptae quaestiones per certa problemata resolvi et in apertum produci possunt, triplici, qua fieri potest, methodo ... ; Accedunt tabulae pro negocio hoc trigonico ... / studio & opera Georgii Ludovici Frobenii Iphoviensis Franci, nunc civis Hamburgensis " **in which all examples are calculated using the common method as well as logarithms and prosthaphaeresis even though he recommends in his own opinion the use of logarithms, and warns against the use of prosthaphaeresis if functions other than sines should arise.**	**vB203**
1573	Seqvvntvr Igitvr Nvnc Canones Tvm Mediorvm seu aequalium motuum, tum **Prosthaphaereseon**: deniq[ue] alij Canones quorum Catalogus supra recitatus est; Reinhold, Erasmus 1573	University library Goettingen
1610	Tabvlae Arithmeticae **Prosthaphaireseōs** Vniversales : Qvarvm Svbsidio Nvmervs Qvilibet, Ex Mvltiplicatione Prodvcendvs, Per Solam Additionem: Et Qvotiens Qvilibet, E Divisione eliciendus, per solam subtractionem, sine taediosâ & lubricâ Multiplicationeis, atque Diuisionis operatione, etiam ab eo, quie Arithmetices non admodum sit gnarus, exactè, celeriter & nullo negotioinuenitur / E' Mvseo Ioannis Georgii Herwart Ab Hohenbvrg. … Authors: Herwarth von Hohenburg, Johann Georg 1610	University library Goettingen
1822	Charles Hutton mentions in the introduction of the 5th edition of his Mathematical Tables (P. 4) John Werner of Nuremberg (1468-1528) as being the author of five books about triangles.	

Summary

In 1901 Axel Anthon Björnbo was working in the Vatican with the collected manuscripts “Joannis Verneri Norimbergensis de triangulis sphaericis” and “Joannis Verneri Norimbergensis de meteoroscopiis” in six books. He stated that this complete work (a copy) was copied by a mathematically uneducated scribe from a - probably hard to read - source.

It is certain that this is not one of Werner’s original manuscripts. However the confirmed contents list of this work confirms the assumption (v. Braunmühl, Heilbronner, Montuela, Eneström) that Johannes Werner is the father of the prosthaphaeresis method. vB141 footnote 1.

Werner is identified as the inventor of the prosthaphaeretic formula! He has certainly worked on it before-hand (1502 - possibly even 1495). However his work was never published, but fell into the hands of others (e.g. Rheticus) as a manuscript in a state which was disordered, contained errors, and was incomplete and illegible.

It is probable that Werner based his work on the developed (and available) work of Regiomontanus, without giving citations from it however.

Björnbo assumes that everything which Regiomontanus has gathered, Werner also admits. B172

"Already via Ibn Yunus (950 - 1009) - publisher of the hakimitic tables - we have found an application of the second of these formulae, and we do not doubt that the Arabs knew how to use it in various ways, although it is not possible for us to provide a direct proof for that opinion.

In no way however did Werner gain his knowledge of the Prosthaphaeretic Methods from him (Ibn Yunus), in that information on the subject is not to be found in those Arab writings which were well-known at that time. Therefore we must regard him as an independent inventor of the Prosthaphaeretic Methods which were so much used in the West."

The above statement concerning Ibn Yunus is in fact not correct, as D.A. King mentions on page 361: "This assertion is incorrect and it can be traced to Delmabre´s misunderstanding of the material in Chapter 15 of the Hakim Zij (the hakimitic tables)."

9 Contemporaries of Johannes Werner

and later scientists associated with prosthaphaeresis

Name	Dates
Bernhard Walther	1430 - 1504
Johannes Müller / Regiomontan	1436 - 1476
Sebald Schreyer	1446 - 1520
Lorenz Beheim	1457 - 1521
Johannes Stabius	1460 - 1522
Erasmus Topler	1462 - 1512
Johannes Werner	1468 - 1522
Willibald Pirckheimer	1470 - 1530
Johannes Schöner	1477 - 1547
Christoph Scheurl II	1481 - 1542
Georg Hartmann	1489 - 1564
Peter Apian	1495 - 1552
Erasmus Reinhold	1511 - 1553
Georg Joachim Rheticus	1514 - 1576
Konrad Gessner	1516 - 1576
Kaspar Wolf	1525 - 1601
Bartholomäus Scultetus/Schulz	1532 - 1614
Christoph Clavius	1537 - 1602
Johann Richter / Prätorius	1537 - 1616
Tycho Brahe	1546 - 1601
Valentinus Otho	1550 - 1605
John Napier	1550 - 1617
Nicolaus Reimers / Ursus	1551 - 1600
Jost Bürgi	1552 - 1632
Jakob Christmann	1554 - 1613
Paul Wittich	1555 - 1587
Melchior Jöstel	1559 - 1611
Christian Severin Longomontan	1562 - 1647
Georg Ludwig Frobenius	1566 - 1645
Johannes Kepler	1571 - 1630

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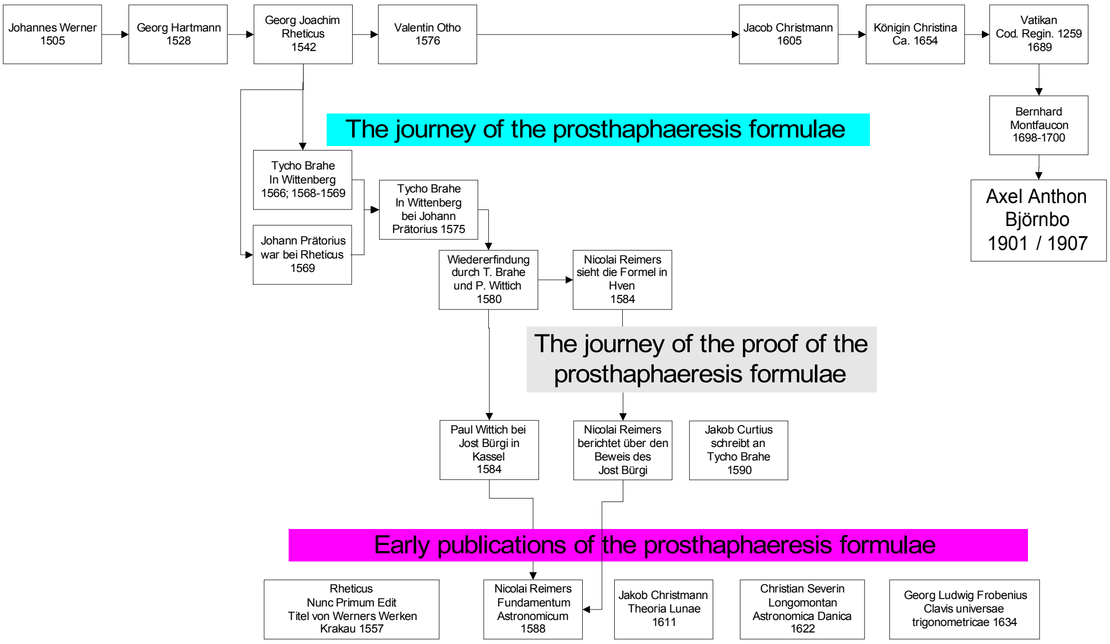
11 The journey through history of the manuscript "De Triangulis Sphaericis Libri Quatuor"



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