2.2 Magic with complex exponentials

We don't really know what aspects of complex variables you learned about in high school, so the goal here is to start more or less from scratch. Feedback will help us to help you, so let us know what you do and don't understand. Also, if something is not immediately clear you should work through examples ... as usual.

The introduction to square roots in school often makes the point that the square root of a negative number is not defined, since after all when we square a number we always get something positive. Then at some point you are told about imaginary numbers, where the basic object is $i = \sqrt{-1}$. It is not clear, perhaps, whether this is some sort of joke (calling them "imaginary" probably doesn't help!). Here we are asking you to take these things very seriously.

Remember that when you first learned about negative numbers (a long time ago ...) there was some mystery about what you do when you add, multiply, etc.. In the end the answer is that the rules are the same, and you have to apply them in a consistent way. This is true also for complex or imaginary numbers.

We begin by recalling that with x and y real numbers, we can form the complex number $z = x + iy$. The object i is the square root of negative one, $i = \sqrt{-1}$. Then if we have two of these numbers

$$
\begin{align}
z_1 &= x_1 + iy_1 \tag{2.45} \\
z_2 &= x_2 + iy_2 \tag{2.46}
\end{align}
$$

we can go through all the usual operations of arithmetic:

$$
\begin{align}
z_1 + z_2 &\equiv (x_1 + iy_1) + (x_2 + iy_2) \tag{2.47} \\
&= (x_1 + x_2) + i(y_1 + y_2); \tag{2.48} \\
z_1 - z_2 &\equiv (x_1 + iy_1) - (x_2 + iy_2) \tag{2.49} \\
&= (x_1 - x_2) + i(y_1 - y_2); \tag{2.50} \\
(z_1) \times (z_2) &\equiv (x_1 + iy_1) \times (x_2 + iy_2) \tag{2.51} \\
&= x_1 x_2 + x_1(iy_2) + iy_1 x_2 + iy_1(iy_2) \tag{2.52} \\
&= x_1 x_2 + i(x_1 y_2 + x_2 y_1) + (i^2)y_1 y_2 \tag{2.53} \\
&= x_1 x_2 + i(x_1 y_2 + x_2 y_1) - y_1 y_2 \tag{2.54} \\
&= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1), \tag{2.55}
\end{align}
$$

where in the second to last step we use the fact that $i^2 = -1$. Note that this list leaves out division, which we'll get back to in a moment.

One very useful operation that is new for complex numbers is called "taking the complex conjugate," or "complex conjugation." For every complex number $z = x + iy$, the complex conjugate is *defined* to be $z^* = x - iy$. Note that in elementary physics we usually use z^* to denote the complex conjugate of z; in the math department and in some more sophisticated physics problems it is conventional to write the complex conjugate of z as $\bar{z}$, but of course this is just notation. The crucial fact is that

$$
\begin{align}
z \times z^* &\equiv (x + iy) \times (x - iy) \tag{2.56} \\
&= x^2 + x(-iy) + iyx + (i)(-i)y^2 \tag{2.57} \\
&= x^2 + i(-xy + yx) - (i^2)y^2 \tag{2.58} \\
&= x^2 + y^2. \tag{2.59}
\end{align}
$$

Often we write $zz^* = |z|^2$, just the way we write the length of a vector in terms of its dot product with itself, $\vec{x} \cdot \vec{x} = |\vec{x}|^2$. This is an important thing on its own, as we will see, but also it makes division a lot easier, which we do now.

There is a trick, which is to clear the complex numbers from the denominator any time we divide:

$$
\begin{align}
\frac{z_1}{z_2} &\equiv \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \tag{2.60} \\
&= \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} \tag{2.61} \\
&= \frac{z_1 z_2^*}{z_2 z_2^*} \tag{2.62} \\
&= \frac{z_1 z_2^*}{|z_2|^2} \tag{2.63} \\
&= \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2} \tag{2.64}
\end{align}
$$

Problem 29: You should be able to add, subtract, multiply and divide these pairs of complex numbers: (a) $z_1 = 3 + 4i$, $z_2 = 4 + 3i$. (b) $z_1 = 3 + 4i$, $z_2 = 4 - 3i$. (c) $z_1 = 7 - 9i$, $z_2 = 27 + 12i$. And you should be able to make up your own examples!

It is useful to think about a complex number as being a vector in a two dimensional space, as in Fig. 2.3. In this view, the x axis is the real part and the y axis is the imaginary part, as is hinted when we write $z = x + iy$. The length of the vector is

$$|z| \equiv \sqrt{x^2 + y^2} = \sqrt{|z|^2} = \sqrt{zz^*}, \tag{2.65}$$

and the angle that this makes with the x axis is given by

$$\theta = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.66}$$

In this notation,

$$z \equiv x + iy \tag{2.67}$$

$$= \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}}\right) \tag{2.68}$$

$$= |z|(\cos\theta + i\sin\theta). \tag{2.69}$$

Now there is a very pretty thing, which is that if we multiply two complex numbers, the magnitudes get multiplied and the angles just add:

$$z_1 \times z_2 \equiv |z_1|(\cos\theta_1 + i\sin\theta_1) \times |z_1|(\cos\theta_1 + i\sin\theta_1) \tag{2.70}$$

$$= (|z_1||z_2|)[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$\quad + i(\sin\theta_1\cos\theta_2 + \sin\theta_1\cos\theta_1)] \tag{2.71}$$

$$= (|z_1||z_2|)[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \tag{2.72}$$

where in the last step we use the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \tag{2.73}$$

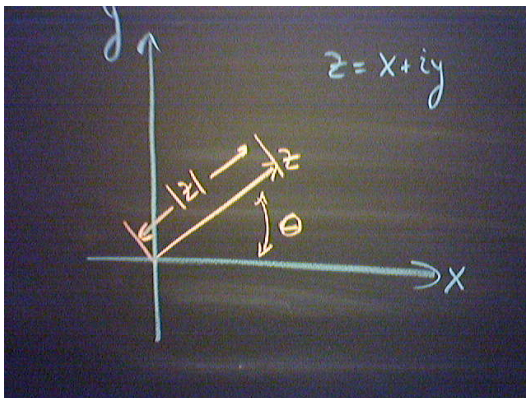$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_1\cos\theta_1. \tag{2.74}$$

Figure 2.3: Thinking of a complex number $z = x+iy$ as a vector in the $x-y$ plane. This is often called the "complex plane."

By the same reasoning, one finds

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \tag{2.75}$$

Problem 30: Derive Eq (2.75). Also, in terms of these θ_1 and θ_2, what is the condition for multiplying two complex numbers and getting a real answer?

We now have enough tools to figure out what we mean by the exponential of a complex number. Specifically, let's ask what we mean by $e^{i\phi}$. This is a complex number, but it's also an exponential and so it has to obey all the rules for the exponentials. In particular,

$$e^{i\phi_1} e^{i\phi_2} = e^{i(\phi_1+\phi_2)} \tag{2.76}$$

$$\frac{e^{i\phi_1}}{e^{i\phi_2}} = e^{i(\phi_1-\phi_2)}. \tag{2.77}$$

You see that the variable ϕ behaves just like the angle θ in the geometrial representation of complex numbers. Furthermore, if we take the complex number $z = e^{i\phi}$ and multiply by its complex conjugate ...

$$e^{i\phi} \times \left[e^{i\phi}\right]^* = e^{i\phi} \times \left[e^{-i\phi}\right] = e^{i(\phi-\phi)} = 1. \tag{2.78}$$

Thus $z = e^{i\phi}$ is a complex number with unit magnitude, and the angle in the complex plane is just ϕ itself. Thus we see that

$$e^{i\phi} = \cos\phi + i\sin\phi, \tag{2.79}$$

which finally tells us what we mean by the complex exponential. Notice that something like this formula had to be true because we know that the solution to the differential equation for the harmonic oscillator can be written either in terms of sines and cosines or in terms of complex exponentials; since solutions are unique, these must be ralted to each other.

If you consider the special case of $\phi = \pi$, then $\sin\phi = 0$ and $\cos\phi = -1$, leading to the famous Euler formula

$$e^{i\pi} + 1 = 0. \tag{2.80}$$

This is a really beautiful equation, linking the mysterious transcendental numbers e and π with the imaginary numbers.

Problem 31: Derive the sum and difference angle identities by multiplying and dividing the complex exponentials. Use the same trick to derive an expression for $\cos(3\theta)$ in terms of $\sin\theta$ and $\cos\theta$.

Armed with these tools, let's get back to our (complex) expression for the trajectory,

$$x(t) = A\exp(+i\omega t) + B\exp(-i\omega t). \tag{2.81}$$

We now know that

$$\exp(\pm i\omega t) = \cos(\omega t) \pm i\sin(\omega t), \tag{2.82}$$

so at least it's clear what our expression means.

To really solve the problem we need to match the initial conditions. We can see that

$$\begin{aligned} x(0) &= A\exp(i\omega \cdot 0) + B\exp(-i\omega \cdot 0) & (2.83)\\ &= A + B, & (2.84) \end{aligned}$$

because $e^0 = 1$, as always. Now in principle A and B are complex numbers,

$$\begin{aligned} A &= \mathrm{Re}A + i\mathrm{Im}A & (2.85)\\ B &= \mathrm{Re}B + i\mathrm{Im}B, & (2.86) \end{aligned}$$

while of course $x(0)$ is the actual position of an object and thus has to be a real number. Let's substitute and see how this works:

$$\begin{aligned} x(0) &= A + B \\ &= \mathrm{Re}A + i\mathrm{Im}A + \mathrm{Re}B + i\mathrm{Im}B & (2.87)\\ &= (\mathrm{Re}A + \mathrm{Re}B) + i(\mathrm{Im}A + \mathrm{Im}B). & (2.88) \end{aligned}$$

So we can match the reality of the initial condition (never mind its value!) only if

$$\mathrm{Im}B = -\mathrm{Im}A. \tag{2.89}$$

Now we need to do the same thing for the initial velocity. By differentiating we see that

$$v(t) \equiv \frac{dx(t)}{dt} = A\frac{de^{i\omega t}}{dt} + B\frac{de^{-i\omega t}}{dt} = A(i\omega)e^{i\omega t} + B(-i\omega)e^{-i\omega t}, \quad (2.90)$$

and hence

$$v(0) = i\omega(A - B). \tag{2.91}$$

Substituting once again,

$$
\begin{aligned}
v(0) &= i\omega(A - B) \\
&= i\omega(\mathrm{Re}A + i\mathrm{Im}A - \mathrm{Re}B - i\mathrm{Im}B) & (2.92) \\
&= i\omega(\mathrm{Re}A\mathrm{Re}B) + (i)(i\omega)(\mathrm{Im}A - \mathrm{Im}B) & (2.93) \\
&= -\omega(\mathrm{Im}A - \mathrm{Im}B) + i\omega(\mathrm{Re}A - \mathrm{Re}B). & (2.94)
\end{aligned}
$$

Notice that the real part of the velocity actually comes from the imaginary parts of A and B. In order that the imaginary part of the velocity cancel must have

$$\mathrm{Re}A = \mathrm{Re}B. \tag{2.95}$$

Thus there really is only one independent complex number here, since we have shown that

$$
\begin{aligned}
A &= \mathrm{Re}A + i\mathrm{Im}A & (2.96) \\
B &= \mathrm{Re}A - i\mathrm{Im}A. & (2.97)
\end{aligned}
$$

When two complex numbers have this relationship—equal real parts and opposite imaginary parts—we say that they are complex conjugates, and the notation for this is $B = A^*$. The operation * simply replaces i by $-i$ in a complex number, and clearly $(z^*)^* = z$. Hence we can write our solution

$$
\begin{aligned}
x(t) &= A\exp(+i\omega t) + B\exp(-i\omega t) \\
&= A\exp(+i\omega t) + A^*\exp(-i\omega t). & (2.98)
\end{aligned}
$$

But note that $\exp(-i\omega t) = [\exp(+i\omega t)]^*$, so we can write

$$
\begin{aligned}
x(t) &= A\exp(+i\omega t) + A^*[\exp(+i\omega t)]^* & (2.99) \\
&= A\exp(+i\omega t) + [A\exp(+i\omega t)]^*. & (2.100)
\end{aligned}
$$

Now $x(t)$ is the sum of a complex number and its complex conjugate. But when we add a complex number to its complex conjugate, we cancel the imaginary part and double the real part:

$$
\begin{aligned}
z + z^* &= [\text{Re}(z) + i\text{Im}(z)] + [\text{Re}(z) - i\text{Im}(z)] & (2.101) \\
&= 2\text{Re}(z). & (2.102)
\end{aligned}
$$

Thus $x(t)$, according to Eq (2.100) will be real at all times. This is good, of course (!). Interestingly, we didn't actually use this condition—all we did was to be sure that we match the initial conditions, which of course are real. This is sufficient to insure that trajectories are real forever, which is nice.

To proceed further, we recall that all complex numbers can be written as

$$
\begin{aligned}
z &= \text{Re}z + i\text{Im}z & (2.103) \\
&= |z|(\cos\phi + i\sin\phi) & (2.104) \\
&= |z|\exp(i\phi), & (2.105)
\end{aligned}
$$

where z is the magnitude of the complex number and ϕ is its phase,

$$
\begin{aligned}
|z| &= \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2} & (2.106) \\
\phi &= \tan^{-1}\left[\frac{\text{Im}(z)}{\text{Re}(z)}\right]. & (2.107)
\end{aligned}
$$

If we do this rewriting of A,

$$
A = |A|\exp(i\phi_A), \qquad (2.108)
$$

then the trajectory becomes

$$
\begin{aligned}
x(t) &= [|A|\exp(i\phi_A)\exp(+i\omega t)] + [|A|\exp(i\phi_A)\exp(+i\omega t)]^* & (2.109) \\
&= [|A|\exp(+i\omega t + i\phi_A)] + [|A|\exp(+i\omega t + i\phi_A)]^*. & (2.110)
\end{aligned}
$$

Thus

$$
\begin{aligned}
x(t) &= [|A|\exp(+i\omega t + i\phi_A)] + [|A|\exp(+i\omega t + i\phi_A)]^* \\
&= 2\text{Re}[|A|\exp(+i\omega t + i\phi_A)] & (2.111) \\
&= 2|A|\text{Re}[\exp(+i\omega t + i\phi_A)] & (2.112) \\
&= 2|A|\cos(\omega t + \phi_A). & (2.113)
\end{aligned}
$$

So, we are done, except that we have to connect this solution to the initial conditions.

We want to find the arbitrary parameters $|A|$ and ϕ_A in terms of the initial position and initial velocity. Let's just calculate:

$$
\begin{aligned}
x(t = 0) &= 2|A| \cos(\omega(0) + \phi_A) & (2.114) \\
&= 2|A| \cos \phi_A & (2.115) \\
v(t = 0) &= \left. \frac{dx(t)}{dt} \right|_{t=0} & (2.116) \\
&= \left. \frac{d}{dt} \left[2|A| \cos(\omega t + \phi_A) \right] \right|_{t=0} & (2.117) \\
&= \left. -2|A|\omega \sin(\omega t + \phi_A) \right|_{t=0} & (2.118) \\
v(t = 0) &= -2|A|\omega \sin \phi_A & (2.119) \\
\frac{v(t = 0)}{\omega} &= -2|A| \sin \phi_A. & (2.120)
\end{aligned}
$$

So we have two equations, Eq's (2.115) and (2.120), that link our parameters to the initial conditions. To solve these equations, note that if we sum the square of the two equations we have

$$
\begin{aligned}
[x(0)]^2 + \left[\frac{v(0)}{\omega} \right]^2 &= (2|A|)^2 \cos^2 \phi_A + (2|A|)^2 \sin^2 \phi_A & (2.121) \\
&= (2|A|)^2, & (2.122)
\end{aligned}
$$

so that

$$
2|A| = \sqrt{[x(0)]^2 + \left[\frac{v(0)}{\omega} \right]^2}. \tag{2.123}
$$

Similarly, if we take Eq (2.120) and divide by Eq (2.115), we find

$$
\begin{aligned}
\frac{v(0)}{\omega x(0)} &= -\frac{2|A| \sin \phi_A}{2|A| \cos \phi_A} & (2.124) \\
&= \tan \phi_A, & (2.125)
\end{aligned}
$$

and hence

$$
\phi_A = \tan^{-1} \left[\frac{v(0)}{\omega x(0)} \right]. \tag{2.126}
$$

Thus the amplitude $|A|$ of the oscillation is related to the initial position, with an extra contribution from the initial velocity, while the phase depends
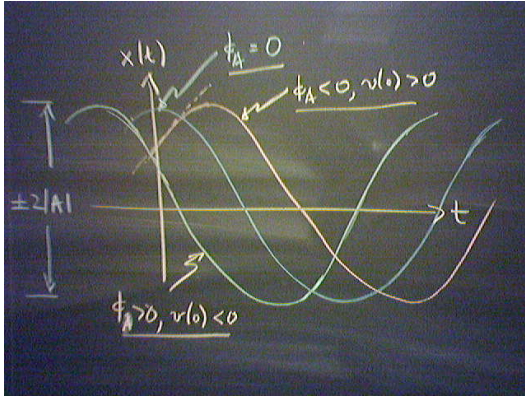
Figure 2.4: Position as a function of time for the harmonic oscillator, from Eq (2.113). Note that when the initial velocity is zero the phase ϕ_A also is zero. Positive initial velocities correspond (in this notation) to negative phases, and vice versa.

on the relative magnitudes of the initial velocity and position. This is shown schematically in Fig 2.4.

Problem 32: Consider the carbon monoxide molecule CO. To a good approximation, the bond between the atoms acts like a Hooke's law spring of stiffness κ and equilibrium length ℓ_0. For the purposes of this problem, neglect rotations of the molecule, so that motion is only in one dimension, parallel to the bond.

(a.) Write the differential equations corresponding to $F = ma$ for the positions x_C and x_O of the two atoms. Remember that the two atoms have different masses m_C and m_O.

(b.) Look for oscillating solutions of the form

$$x_C(t) = x_C^0 + A\exp(-i\omega t) \tag{2.127}$$
$$x_O(t) = x_O^0 + B\exp(-i\omega t), \tag{2.128}$$

where the resting positions x_c^0 and x_O^0 are chosen to match the equilibrium length of the bond. Show that solutions of this form exist, and find the natural frequency ω for these oscillations.