Proofs of Parseval's Theorem & the Convolution Theorem

(using the integral representation of the δ-function)

1 The generalization of Parseval's theorem

The result is

$$\int_{-\infty}^{\infty} f(t)g(t)^*\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)\overline{g}(\omega)^*\,d\omega \tag{1}$$

This has many names but is often called Plancherel's formula.

The key step in the proof of this is the use of the integral representation of the δ-function

$$\delta(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\tau\omega}\,d\omega \qquad \text{or} \qquad \delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm i\tau\omega}\,d\tau\,. \tag{2}$$

We firstly invoke the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)e^{i\omega t}\,d\omega \tag{3}$$

and then use this to re-write the LHS of (1) as

$$\int_{-\infty}^{\infty} f(t)g(t)^*\,dt = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\omega)e^{i\omega t}\,d\omega\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{g}(\omega')^*e^{-i\omega' t}\,d\omega'\right)dt\,. \tag{4}$$

Re-arranging the order of integration we obtain

$$\int_{-\infty}^{\infty} f(t)f(t)^*\,dt = \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f}(\omega)\overline{g}(\omega')^*\underbrace{\left(\int_{-\infty}^{\infty}e^{i(\omega-\omega')t}\,dt\right)}_{Use\ delta-fn\ here}d\omega'\,d\omega\,. \tag{5}$$

The version of the integral representation of the δ-function we use in (2) above is

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\omega-\omega')}\,dt\,. \tag{6}$$

Using this in (5), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)g(t)^*\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\omega)\left(\int_{-\infty}^{\infty}\overline{g}(\omega')^*\delta(\omega-\omega')\,d\omega'\right)d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\omega)\overline{g}(\omega)^*\,d\omega\,.\end{aligned} \tag{7}$$

(7) comes about because of the general δ-function property $\int_{-\infty}^{\infty} F(\omega')\delta(\omega-\omega')\,d\omega' = F(\omega)$.

2 Parseval's theorem (also known as the energy theorem)

Taking $g = f$ in (1) we immediately obtain

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{f}(\omega)|^2\,d\omega\,. \tag{8}$$

The LHS side is energy in temporal space while the RHS is energy in spectral space.

Example: Sheet 6 Q6 asks you to use Parseval's Theorem to prove that $\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^2} = \pi/2$. The integral can be evaluated by the Residue Theorem but to use Parseval's Theorem you will need to evaluate $\overline{f}(\omega) = \int_{-\infty}^{\infty}\frac{e^{-i\omega t}dt}{1+t^2}$. To find this, construct the complex integral $\oint_C \frac{e^{-i\omega z}dz}{1+z^2}$ and take the semi-circle C in the upper (lower) half-plane when $\omega < 0$ (> 0). The answers are πe^{ω} when $\omega < 0$ and $\pi e^{-\omega}$ when $\omega > 0$. Then evaluate the RHS of (8) in its two parts.

3 The Convolution theorem and the auto-correlation function

The statement of the Convolution theorem is this: for two functions $f(t)$ and $g(t)$ with Fourier transforms $\mathcal{F}[f(t)] = \overline{f}(\omega)$ and $\mathcal{F}[g(t)] = \overline{g}(\omega)$, with convolution integral defined by[1]

$$f \star g = \int_{-\infty}^{\infty} f(u) g(t-u)\, du\,, \tag{10}$$

then the Fourier transform of this convolution is given by

$$\mathcal{F}\,(f \star g) = \overline{f}(\omega)\,\overline{g}(\omega)\,. \tag{11}$$

To prove (11) we write it as

$$\mathcal{F}\,(f \star g) = \int_{-\infty}^{\infty} e^{-i\omega t} \left(\int_{-\infty}^{\infty} f(u) g(t-u)\, du \right) dt\,. \tag{12}$$

Now define $\tau = t - u$ and divide the order of integration to find

$$\mathcal{F}\,(f \star g) = \int_{-\infty}^{\infty} e^{-i\omega u} f(u)\, du \int_{-\infty}^{\infty} e^{-i\omega\tau} g(\tau)\, d\tau = \overline{f}(\omega)\,\overline{g}(\omega)\,. \tag{13}$$

This step is allowable because the region of integration in the $\tau - u$ plane is infinite. As we shall later, with Laplace transforms this is not the case and requires more care.

The normalised auto-correlation function is related to this and is given by

$$\gamma(t) = \frac{\int_{-\infty}^{\infty} f(u) f^{*}(t-u)\, du}{\int_{-\infty}^{\infty} |f(u)|^{2}\, du}\,. \tag{14}$$

[1] It makes no difference which way round the f and the g inside the integral are placed: thus we could write

$$f \star g = \int_{-\infty}^{\infty} f(t-u) g(u)\, du\,. \tag{9}$$